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MULTIPLE SCATTERING OF WAVES BY PLANAR RANDOM DISTRIBUTIONS OF PARALLEL CYLINDERS AND BOSSES

by

VIC TWERSKY

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Morris Kline
Project Director

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Abstract

The two-dimensional problem of multiple scattering of a plane wave by a planar 'random' distribution of parallel cylinders is considered. A formal solution, ψ, for the wave scattered by a single configuration is obtained, and then ψ and $|\psi|^2$ are averaged over an ensemble of configurations specified by the probability distribution function for pairs of cylinders. (The treatment is general enough to include both the limiting cases of either a periodic or a 'uniform' distribution.) Closed-form approximations for the averages are derived and used to obtain the average energy flux and to discuss certain energy theorems. The results are then applied to the scattered reflection of electromagnetic or acoustic waves from striated surfaces composed of cylindrical bosses on a perfectly reflecting plane. It is shown, for example, that whereas the single-scattered wave may become infinite as grazing incidence is approached, the multiple-scattered wave approaches the negative of the incident wave; thus the results reduce to those for a dielectric plane near grazing incidence. The maximum effects of multiple scattering are discussed, and practical expressions for the reflection coefficients and differential scattering cross sections are obtained.

1. Introduction

We consider the two-dimensional problem of multiple scattering of a plane wave by a planar 'random' distribution of parallel cylinders. Initially we present a formal solution, say ψ, for the wave scattered by a single configuration of parallel cylinders. Then we average ψ and $|\psi|^2$ over an ensemble of configurations specified by the probability distribution function for pairs of cylinders; here we use the function derived by Zernike and Prins[1] in their single-scattering treatment of X-ray diffraction by liquids. We derive closed-form approximations for the averages of ψ and $|\psi|^2$ and use these to determine the average energy flux and to discuss certain energy theorems. The results are then applied to the scattered reflections of electromagnetic and acoustic waves from striated surfaces composed of cylindrical bosses on a perfectly reflecting plane[2].

Since we previously derived a 'successive series' representation[3] for ψ, we could now perform the averages and seek approximations in closed form. For simplicity, however, we follow rather a 'self-consistent field approach' similar to that employed by Reiche[4], Born[5],

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1. F. Zernike and J.A. Prins, Zeits, Physik, 41, 184(1927).
2. V. Twersky, J. Acoust. Soc. Am., 22, 539(1950); 23, 336(1951); J. Appl. Phys. 22, 825(1951). These papers contain an approximate treatment of a 'uniform' distribution of scatterers whose radii, a, are small compared to $k^{-1} = \lambda/2\pi$. Our present analysis shows that the previous results are correct to the order of $(ka)^2$ except in the vicinity of grazing incidence.
3. V. Twersky, J.Acoust. Soc. Am. 24, 42(1951); J. Appl. Phys., 23, 407 (1952) and 1099(1952). The first paper deals with an arbitrary configuration of parallel cylinders; the second deals with a planar configuration and with two cylinders; the third deals with the grating. Here the scattered wave is expressed as an infinite series of 'orders of scattering'; the first order, the single-scattering approximation, results from the excitation of each cylinder by only the incident plane wave; the second order results form the excitation of each by the first order of scattering from the remaining cylinders, etc. Henceforth we refer to this information as 'successive scattering' and use 'multiple scattering' to indicate any rigorous approach to a many-body scattering problem.
4. F. Reiche, Ann. Physik, 50, 1 and 121(1916); see Section 8.
5. M. Born, Optik(Springer, Berlin, 1933), p. 313 ff.

Foldy[6] and Lax[7] for random distributions and by Ewald[8] and Karp[9] for periodic configurations. (We revert to the successive scattering formalism at one point in order to interpret a heuristic approximation.)

We derive closed-form representations which are valid at least when the minimum separation of cylinders is large compared to radius and wavelength; these apparently suffice to take into account the largest effects of multiple scattering. We show that these representations hold either when there is a high degree of order in the distribution of neighbors, or for all distributions near grazing incidence. Other representations are also discussed.

The most interesting result we obtain concerns the behavior near grazing incidence. Whereas the single-scattered wave may become infinite as grazing incidence is approached, the multiple-scattered wave approaches the negative of the incident wave. Thus the results reduce to those for a dielectric plane near grazing incidence, the incident wave being perfectly reflected with the appropriate sign change.

Expressions for the reflection coefficients and for the differential scattering cross sections are derived and applied to the case of radius small compared to wavelength.

We are grateful to Professors J.B. Keller, B. Friedman and F. Reiche for many fruitful discussions and practical aid.

2. Preliminary Considerations and Statement of the Problem

This section is divided into three parts: in the first part we present a formal solution for the wave scattered by a single configurations of parallel cylinders; in the second we define an ensemble of configurations; in the third we state the problem of this paper explicitly.

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6. L.L. Foldy, Phys. Rev., 67, 107(1945). This excellent paper and the following papers by Lax should be referred to for discussion of many points we but touch on.
7. M. Lax, Revs. Mod. Phys., 23, 287(1951); Phys. Rev. 80, 621(1952).
8. P.P. Ewald, Ann. Physik, 49, 1 and 117(1916).
9. Private communication. This work also was presented at the Symposium on Microwave Optics, McGill University, June, 1953.

2.1 Scattering by a Planar Configuration of Parallel Cylinders

We first consider the two-dimensional problem of the scattering of a plane electromagnetic or acoustic wave by a planar configuration of N parallel identical cylinders of radius a. The arbitrarily spaced cylinders are located at $x = 0$, $y = y_s$. The observation point is located at r, θ or at r_s, θ_s with respect to cylinder s; see Fig. 1.

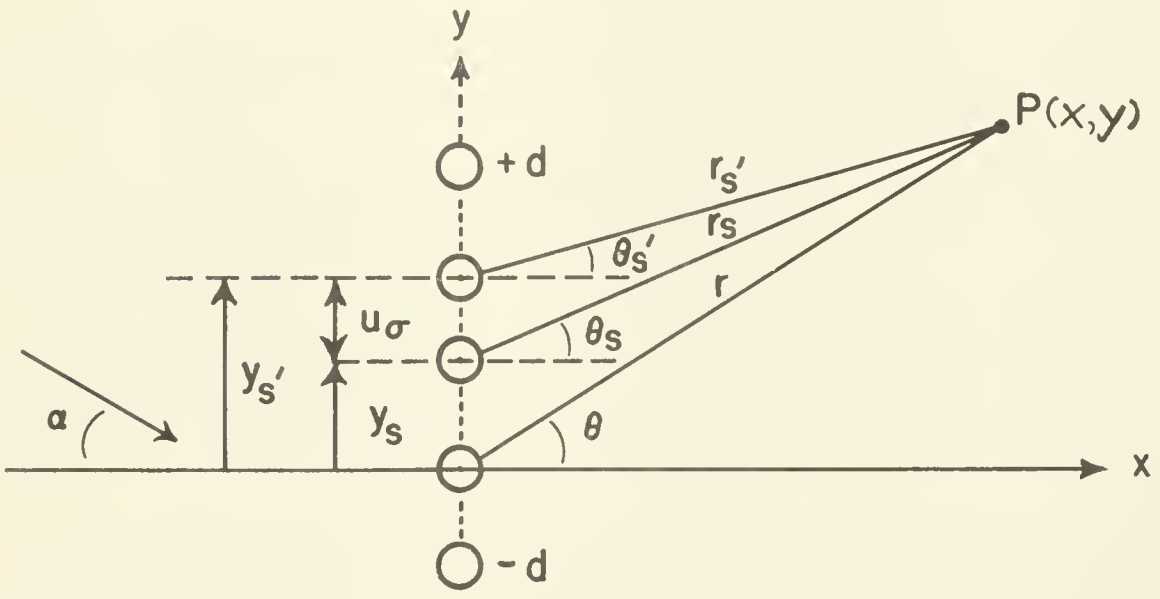

Fig. 1. Plane wave incident on a planar configuration of N parallel cylinders.

We require a solution of the reduced wave equation $(\nabla^2+k^2)\,\Psi = 0$ for $k = 2\pi/\lambda$ which satisfies prescribed boundary conditions at the surfaces $r_s = a$ of all cylinders; the suppressed time factor is $\exp(-i\omega t)$. The function Ψ corresponds to one of the Hertz potentials for electromagnetics or to the velocity potential for acoustics[10].

10. See V. Twersky, Research Report EM-34, Mathematics Research Group, Washington Square College, NYU, 1951, for further discussion of the above and certain of the following manipulations.

We specify that the solution be of the form

$$(1) \qquad \Psi = \psi_i + \sum_s \psi_s = \psi_i + \psi \ ,$$

where ψ_i is a plane wave incident at an angle $-\alpha$, and where ψ_s, the wave scattered by cylinder s, satisfies Sommerfeld's radiation condition as $r_s \to \infty$; ψ is the total scattered wave.

We expand the incident wave in terms of cylindrical waves referred to the axis of cylinder s:

$$(2) \quad \psi_i = \exp\left[ikr\cos(\theta + \alpha)\right] = \exp(-iky_s\sin\alpha)\exp\left[ikr_s\cos(\theta_s + \alpha)\right]$$

$$= \exp(-iky_s\sin\alpha) \sum_{n=-\infty}^{\infty} J_n(kr_s) i^n \exp\left[in(\theta_s + \alpha)\right] \ ,$$

where the J_n are the Bessel functions. The wave scattered by cylinder s may be expanded in terms of outgoing cylindrical waves referred to the axis of cylinder s:

$$(3) \quad \psi_s = \exp(-iky_s\sin\alpha) \sum_n B_{ns}\, i^n H_n(kr_s) \exp(in\theta_s), \ H_n = H_n^{(1)},$$

where the H_n are Hankel functions of the first kind. The scattering coefficients B_{ns}, which are in general functions of the positions of all cylinders (i.e., of the configuration), are to be determined from the boundary conditions.

In order to impose the boundary conditions at $r_s = a$, we must expand $\psi_{s'}$, $s' \neq s$, in terms of cylindrical waves referred to the origin of cylinder s. Making use of the 'addition

theorem' for cylindrical waves[11], we have at $r_s = a$

$$\psi_{s'} = \exp(-iky_{s'}\sin\alpha) \sum_{m=-\infty}^{\infty} B_{ms'} i^m H_m(kr_{s'}) \exp(im\theta_{s'})$$

(3')

$$= \exp(-iky_{s'}\sin\alpha) \sum_n \exp(in\theta_s) i^n J_n(ka) \sum_m B_{ms'} H_{n-m}(k|y_s - y_{s'}|) \exp\left[i(n-m)(\tfrac{\pi}{2} - \theta_{s's})\right],$$

where $\theta_{s's} = \pm \pi/2$ for $s' \gtrless s$.

From (1) we write $\Psi = \psi_i + \psi_s + \sum_{s'}{}' \psi_{s'}$, where the prime on the sum indicates that $s' \neq s$. Subsituting (2), (3) and (3') into (1) we see that at $r_s = a$, we may write $\Psi = \sum_n \exp(in\theta_s)\, D_n(s)$, where the D are independent of θ_s. Hence on imposing the boundary condition $\Psi = 0$ at $r_s = a$, we may equate the coefficients of $\exp(in\theta_s)$ to zero and thus obtain the scattering coefficients

$$(4)\quad B_{ns} = A_n \left\{ e^{in\alpha} + \sum_{s'}{}' \exp\left[ik(y_s - y_{s'})\sin\alpha\right] \sum_m H_{n-m}(k|y_s - y_{s'}|) B_{ms'} \exp\left[i(n-m)(\tfrac{\pi}{2} - \theta_{s's})\right] \right\}.$$

where A_n represents $-J_n(ka)/\, H_n(ka)$, the single-scattered value of the coefficient. (Here Ψ is the z-component of the electric field for the case of perfectly conducting cylinders.)

Similarly for the boundary condition $\frac{\partial \Psi}{\partial r_s} = 0$ at $r_s = a$, we obtain (4) with A_n equal to $-J_n'(ka) \,/\, H_n'(ka)$, $J'(x) = \frac{\partial J}{\partial x}$; Ψ is now the z-component of the magnetic field for perfectly conducting cylinders.

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11. G.N. Watson, Bessel Functions (Cambridge Press, 1948) pp. 359-361.

For the more general continuity boundary conditions we must also take into account the wave within each cylinder; we write this wave as

$$(4') \qquad \Phi_s = \exp(-iky_s \sin\alpha) \sum_n D_{ns} i^n J_n(k'r_s) \exp(in\theta_s) ,$$

where k', which is in general complex, is the propagation number within the cylinder. Imposing the continuity boundary conditions

$$(\Psi - M\Phi_s)_{r_s=a} = 0 \quad , \quad \left[\frac{\partial}{\partial r_s} (\Psi - K\Phi_s) \right]_{r_s=a} = 0 ,$$

where Ψ is as in (1) and where M and K are constants, we obtain

$$B_{ns} = A_n \left\{ \quad \right\} , \quad D_{ns} = E_n \left\{ \quad \right\} ,$$

where the curly brackets represent the function in curly brackets in (4), and where

$$A_n = -\left[Kk' J_n(ka) J_n'(k'a) - Mk J_n'(ka) J_n(k'a) \right] D^{-1} ,$$

$$E_n = -k\left[H_n'(ka) J_n(ka) - H_n(ka) J_n'(ka) \right] D^{-1} ,$$

$$D = Kk' H_n(ka) J_n'(k'a) - Mk H_n'(ka) J_n(k'a) .$$

The coefficients A_n and E_n are the scattering and 'transmission' coefficients respectively of the isolated cylinder. The constants M and K have the following values:

if the incident wave is polarized parallel to the cylinder(TM case), then $M = 1$, $K = \mu / \mu' =$ ratio of the permeabilities; if the incident wave is polarized perpendicular to the cylinder (TE case), then $M = k' \mu / k \mu'$, $K = k/k'$; for the acoustic case $M = g'/g$ = ratio of densities, $K = 1$. The above correspond to the physical requirements that at $r_s = a$, either E_{tan} and H_{tan} or the normal particle velocity and excess pressure be continuous. (See ref. 10, sec. 1.6 for a detailed discussion.) It can be seen from the above that the determination of B_{ns} yields D_{ns} as well; we therefore restrict discussion to the first set of coefficients in the major part of this paper.

Eq. (4) defines the scattering coefficients B_{ns} in terms of an infinite set of linear equations. On solving these equations by iteration, i.e., by successive substitutions for B_{ms} , , we obtain the infinite series of 'orders of scattering' derived directly in ref. 3; this series, if it converges, constitutes the solution to the problem. (From physical considerations, we believe that this exact series converges in general.) If the series converges very rapidly, then we need retain only the leading terms; however, since this case corresponds to the effects of multiple scattering being very small, it is of interest primarily in determining the range of validity of the single-scattering approximations. (We use this series in sec. 3.3 to investigate a certain heuristic approximation.)

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If the series does not converge rapidly, then it is simpler to work directly with (4) or one of its approximate representations. There are at least two methods of approximation available:

1) We may reduce (4) to a set of only ν equations if we assume that only the first ν B_{ns} are non-vanishing. This procedure is plausible provided that the radius is small compared to wavelength, i.e., the A_n behave essentially like $(ka)^{2n} / (n!)^2$ for $ka \longrightarrow 0$ (A_0 may behave like $1/\log ka$). Thus if we assume that only A_0 is non-vanishing,(i.e., the case of 'isotropic scatters'), then we may approximate (4) by

$$(5) \qquad B_{os} \approx A_o \left[1 + \sum_{s'}{}' \exp\left[ik(y_s - y_{s'})\sin\alpha \right] H_o(k|y_s - y_{s'}|)\, B_{os'} \right] ,$$

which consists of N equations for the N unknown B_{os}.

2) We may reduce (4) to a set of 2N equations if we assume that each cylinder is in the far fields of its neighbors. i.e., if we replace H_{n-m} by the leading term of its asymptotic expansion for argument large compared to order:

$$(6) \qquad H_{n-m}(ku) \sim i^{-n+m} (2/i\pi ku)^{1/2} e^{iku} \equiv i^{-n+m} H(u) .$$

If we now substitute (6) into (3) and (4) and introduce the 'scattering amplitudes'

$$(7) \qquad G_s(\theta_s, -\alpha) \equiv \sum_n B_{ns} \exp(in\theta_s), \quad C_s(\theta_s, -\alpha) \equiv \sum_n A_n \exp\left[in(\theta_s + \alpha) \right] ,$$

obtain

$$\psi = \sum \psi_s \sim \sum_s \exp(-iky_s \sin\alpha)\, H(r_s)\, G_s(\theta_s, -\alpha) ,$$

$$G_s(\theta_s, -\alpha) \sim C(\theta_s, -\alpha) + \sum_{s'}{}' C(\theta_s, \theta_{s's}) \exp\left[ik(y_s - y_{s'})\sin\alpha \right] H(|y_s - y_{s'}|)\, G_{s'}(\theta_{s's}, -\alpha) .$$

physical interpretation of $G_s(\theta_s, -\alpha)$ is as follows: the term $C(\theta_s, -\alpha)$ the far-field response of an isolated cylinder s to a plane wave ident at an angle $-\alpha$. The remaining terms indicate the response to cylindrical waves reaching s from cylinders $s' \gtrless s$ at angles

$\theta_{s's} = \mp \pi/2$. The amplitude of these cylindrical waves is simply $G_{s'}(\theta_{s'},-\alpha)$ evaluated in the plane of the configuration. Hence substituting $\theta_s = \pm \pi/2$ into (8) yields 2N equations for the 2N unknowns $G_s(\pm \pi/2, -\alpha)$. Eq. (8), which is not necessarily restricted to circular cylinders, could have been written from elementary physical considerations. An expression of this form was first obtained by Karp[9] for the grating.

The procedure that yielded (8) is plausible provided there exists a minimum separation of neighbors, $u_{min} \equiv b$, such that $kb \gg 1$ and $b \gg a$. This requirement is deduced from the known behavior of the single cylinder. For example, if the field vanishes on the cylinder, then only the first $n \approx ka + 2$ terms of the solution for the isolated cylinder are important. Hence we must assume that the arguments of the Hankel functions are large compared to ka and large compared to unity in order to employ (6).

For simplicity we use (8) rather than the rigorous expressions (3) and (4) in the major part of this paper. However, we are able to show in some cases that results obtained from (8) differ only negligibly from those obtained from the rigorous expressions.

2.2 The Ensemble of Configurations

The total scattered wave in eq. (1) may be written as

$$\psi = \sum_{s=1}^{N} \psi_s(y_s; y_1, y_2 \ldots\ldots y_N) \tag{9}$$

to indicate that the wave scattered by each cylinder is an explicit function of the positions of all cylinders, i.e., of the configuration specified by a particular set of values for the position variables y_1 to y_N. In general we specify an ensemble of configurations by a pro-

bability distribution function $W_N(y_1, y_2 \ldots\ldots y_N)$ such that $W_N dy_1 \ldots\ldots dy_N$ is the probability of finding the N cylinders in a configuration in which cylinders 1 to N are in the elements of range dy_1 to dy_N (i.e., cylinder 1 is between y_1 and $y_1 + dy_1$, etc.). We average ψ and $|\psi|^2$ over the ensemble by multiplying by W_N and integrating with respect to $y_1, \ldots\ldots y_N$ over the range available to each cylinder. (See Foldy[6] p. 109 and Lax[7] p. 299 for further discussion of the probability features of a many-body scattering problem.)

We note that in a single-scattering treatment (i.e., when each cylinder is treated as isolated from the others), ψ_s depends only on y_s while $\psi_s \psi_{s'}^*$ (where * indicates the complex conjugate) depends only on $y_s, y_{s'}$; consequently, for the averaging, only the probability distribution $W_1(y_s)$ for a single cylinder and the joint probability distribution $W_2(y_s, y_{s'})$ for pairs of cylinders are required. W_2 and W_1 are obtained by integrating W_N over the remaining N - 2 and N - 1 variables, respectively.

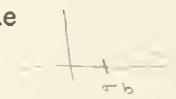
The ensemble we consider is such that W_N is completely specified by $W_2(y_s, y_{s'}) = W_1(y_s)\, P(y_s|y_{s'})$, where $P(y_s|y_{s'})\, dy_{s'}$ is the conditional probability that given s in dy_s one finds s' in $dy_{s'}$. Thus $W_3(y_s, y_{s'}, y_{s''})$, which may in general be written as $W_1(y_s)\, P(y_s|y_{s'})\, P(y_s, y_{s'}|y_{s''})$, reduces to $W_1(y_s)\, P(y_s|y_{s'}) P(y_{s'}|y_{s''})$, etc. (i.e., we neglect correlations higher than those between pairs). We assume

$$(10) \quad W_1(y_s) = (2d)^{-1} \equiv \rho/N \quad , \quad P(y_s|y_{s'}) = p_{|s-s'|}(|y_s - y_{s'}|) \equiv p_\sigma(u_\sigma) \, ,$$

where 2d is the range available to the scatterers and ρ is the average number of cylinders in unit length. The function $p_\sigma(u_\sigma)du_\sigma$ is the probability that

the distance between the axis of any cylinder and that of its σ-th neighbor is between u_σ and $u_\sigma + du_\sigma$. Eq. 10 states that all positions on the available range are equiprobable for the individual cylinders and that the distribution of the neighbors of a given cylinder is independent of that cylinder's position.

Introducing the symbols b and $\bar{b} = \rho^{-1}$ to represent the minimum and average separations of the axes of adjacent cylinders, and using $\delta = \bar{b} - b$, we may write the distribution function for the separation between any cylinder and its σ-th neighbor as



$$(11) \quad p_\sigma(u) = \frac{(u - \sigma b)^{\sigma-1}}{\delta^{\sigma-1}(\sigma-1)!} e^{-(u-\sigma b)/\delta}, \; u \geq \sigma b; \; p_\sigma(u) = 0, \; u < \sigma b;$$

this function is derived in reference 1. (Note that $p_\sigma \delta$ is the probability of finding any $\sigma - 1$ elements of width b in the range $u - \sigma b$, and that du/δ is the probability of finding an additional element between u and u + du.) The distribution function p_σ , governed by the parameter δ/b, is sufficiently general to include the limiting cases of the periodic and 'uniform' distributions.

The total probability of finding one of M neighbors between the distance u and u + du from a given cylinder may be written as p(u) du, where

$$(12) \quad p(u) = \sum_{\sigma=1}^{M} p_\sigma(u).$$

From (11) we see that p(u) is zero for $u < b$, that it has its maximum value δ^{-1} at u = b, and that it is oscillatory for $u > b$. To investigate its behavior for large u, we must derive a more convergent representation. The Laplace transform of p(u) is

$$h(w) = \sum_{\sigma=1}^{M} \int_{\sigma b}^{\infty} p_\sigma(u) e^{-wu}\, du = \sum \frac{e^{-\sigma wb}}{(\sigma-1)!} \int_0^{\infty} e^{-v(1+\delta w)} v^{\sigma-1}\, dv$$

(13)

$$= \sum \left[\frac{e^{-wb}}{1+w\delta}\right]^{\sigma} = \frac{e^{-wb}}{1+w\delta-e^{-wb}} \left[1 - \frac{e^{-Mwb}}{(1+w\delta)^M}\right] .$$

Applying the inverse transform for $M \to \infty$ we express $p(u)$ as a sum of residues:

$$(14)\quad p(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} h(w) e^{wu}\, dw = \sum_{\nu} \frac{\exp(uw_\nu)}{b+\delta\exp(bw_\nu)} , \quad u > b ,$$

where the w_ν are roots of $(1 + \delta w) = e^{-wb}$. The contribution of the pole at the origin, $\nu = 0$, is simply $(\delta + b)^{-1} = \rho$, i.e., the average number of cylinders in unit length of distribution.

If δ/b is large, then we need retain only the terms $\nu = 0, \pm 1$ in (14). We write $w_0 = 0$, $w_{\pm 1} = -\alpha \pm i\beta$, where α and β are real and positive†, and we obtain

$$(15)\quad p(u) \approx \rho + \frac{e^{-\alpha u}\left\{2b\cos\beta u + 2\delta\cos\left[\beta(u-b)\right] e^{-\alpha b}\right\}}{b^2 + 2b\delta\, e^{-\alpha b}\cos\beta b + \delta^2 e^{-2\alpha b}} , \quad u > b \ll \bar{b} .$$

more random

Hence p approaches ρ in an exponentially decreasing oscillatory fashion as u increases; i.e., the distribution of neighbors is 'ordered' only in the immediate vicinity of each cylinder; this corresponds to the so-called 'local order' of liquid state theory. For very large δ/b , p is thus essentially a step function which equals zero for $u < b$ and ρ

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† It has been shown by Prof. Reiche (private communication) that if we write $w_{\pm|\nu|} = \alpha \pm i\beta_{|\nu|}$ for large δ/b, then $a \approx (\gamma/b)(1+1/\gamma)$ and $\beta \approx (2|\nu|-1)(\pi/b)(1+1/\gamma)$ with $\gamma \approx \log\left[(\delta/b)\log(\delta/b)\right]$.

thereafter. In the limit $b \to 0$, (15) reduces to the results for a 'uniform' distribution, i.e., $p(u) = \rho$ for all values of u. We will refer to (15) as characterizing the 'more random' case.

On the other hand if $\delta/b \to 0$, then the roots may be approximated by $w_\nu \approx (2\nu\pi i + \epsilon)/b$. We find $\epsilon = -i2\nu\pi\rho\delta$, $w_\nu \approx i2\nu\pi\rho$, and consequently (14) reduces to

more ordered

$$(16) \qquad p(u) \approx \rho + \sum_{\nu=\pm 1}^{\infty} \frac{e^{i\nu 2\pi\rho u}}{b+\delta e^{i\nu 2\pi\rho b}} \quad , \quad u > b \approx \bar{b} \ .$$

We refer to (16) as the 'more ordered' case. In the limit $\delta \to 0$, (16) reduces to the results for a diffraction grating of spacing b.

2.3 Statement of the Problem

We are now in a position to state explicitly the problem considered in this paper. We are given the wave scattered by a single configuration of cylinders, i.e., $\psi = \sum \psi_s$, where ψ_s as given by (3) and (4) is expressed in terms of the known solution for the single cylinder. We seek to average ψ and $|\psi|^2$ over the ensemble specified by the distribution function $W_2(y_s, y_{s'}) = W_1(y_s)\, P(y_s|y_{s'})$ for pairs where W_1 and P are given by (10) and (11). We also seek closed-form approximations for the averages which may be applied to the physical problem of scattered reflection from a striated surface.

Sections 3 and 4 are concerned with the average values of ψ and $|\psi|^2$ respectively. Section 5 deals with the average energy flux and certain energy theorems. Section 6 treats the analogous problem for bosses on a perfectly reflecting plane.

In the following sections we use either a bar or angular brackets to indicate an average value.

3. The Average Wave Function

The average value of the scattered wave defined by (1), (3) and (4) may be written as

$$\bar{\psi} = \sum_s \bar{\psi}_s = \sum_s \int\cdots\int \psi_s(y_s;\, y_1 \ldots y_N)\, W_N(y_1 \ldots y_N)\, dy_1 \ldots dy_N$$

(17)

$$= \sum_s \sum_n i^n \int_{-d}^{d} W_1(y_s) \exp(-iky_s \sin\alpha) H_n(kr_s) \exp(in\theta_s) < B_{ns} >_s \, dy_s \,,$$

where $< B_{ns} >_s$ indicates that B_{ns} as in (4) is to be averaged over all variables except y_s, i.e., over all configurations assumed by cylinders $s' \neq s$. We refer to $< >_s$ as an average with one variable held fixed. (We use essentially the same notation and nomenclature as Foldy[6] and Lax[7].) Similarly we have

average of (4)

$$< B_{ms's} >_s = A_n \left\{ e^{in\alpha} + \sum_{s'}{}' \sum_m \int_{-d}^{d} \exp\left[ik(y_s - y_{s'})\sin\alpha\right] H_{n-m}(k|y_s - y_{s'}|) \right.$$

(18)

$$\left. \times \exp\left[i(n-m)\left(\frac{\pi}{2} - \theta_{s's}\right)\right] P(y_s|y_{s'}) < B_{ms's} >_{ss'} \, dy_{s'} \right\},$$

where $< B_{ms's} >_{ss'}$ is the average scattering coefficient with two variables held fixed.

We now restrict the parameters so that $d \to \infty$ and $N \to \infty$ while $NW_1(y_s) = N/2d = \rho$ remains constant. As a consequence we have

(19) $$< B_{ns} >_s = < B_{ns's} >_{s'} = \bar{B}_n \,;$$

thus the average scattering coefficient with one variable held fixed is independent of y_s; i.e., no 'end effects' occur for an infinite range. (The result (19), which follows from elementary physical considerations, can be deduced from (18) with $N \to \infty$, $d \to \infty$, if it is kept in mind that $P(y_s|y_{s'})$ as defined in (10) is a function only of $|y_s - y_{s'}|$.) In view of (19) we may sum over s in (17), replace y_s by y', and write

$$(20)\quad \bar{\psi} = \rho \sum_n \bar{B}_n i^n \int_{-\infty}^{\infty} e^{-iky'\sin\alpha} H_n(kr') e^{in\theta'} dy', \qquad r' = \left[x^2 + (y-y')^2\right]^{\frac{1}{2}} = \frac{y-y'}{\sin\theta'} .$$

We reserve discussion of $\bar{B}_n$ for Section 3.1.

The integral over y' can be reduced to one evaluated exactly by Reiche[12]:

$$(21)\quad \int_{-\infty}^{\infty} H_n\left(q\sqrt{1+v^2}\right)\left[\frac{1+iv}{\sqrt{1+v^2}}\right]^n e^{-ivp}\, dv = \frac{2}{\sqrt{q^2-p^2}}\, e^{i\sqrt{q^2-p^2}}\, i^{-n}\left[\sqrt{1-\frac{p^2}{q^2}} + \frac{ip}{q}\right]^n \equiv I_n .$$

Thus if we introduce $v = (y - y')/x$ into (20) and write $\theta' = \tan^{-1} v = \frac{1}{2i}\log\frac{1+iv}{1-iv}$, we obtain

$$\bar{\psi} = \rho e^{-iky'\sin\alpha} \sum \bar{B}_n\, i^n\, I_n ,$$

where I_n is as in (21) with $q = kx$ and $p = -ikx\sin\alpha$. On taking the

12. F. Reiche, Research Report No. EM-61, Mathematics Research Group, Washington Square College, NYU.

positive and negative roots of $\sqrt{1 - p^2/q^2}$ for $x \gtrless 0$ respectively, we obtain

$$xI_n = i^{-n}(2/k\cos\alpha)e^{\mp ikx\cos\alpha}\,(\mp\cos\alpha - i\sin\alpha)^n\ ;$$

consequently $\bar{\psi}$ of (20) reduces to

$$(22)\qquad \bar{\psi} = \frac{2\rho}{k\cos\alpha}\, e^{ik(|x|\cos\alpha - y\sin\alpha)} \sum \bar{B}_n \begin{Bmatrix} e^{-in\alpha} \\ e^{in(\pi+\alpha)} \end{Bmatrix}, \quad \begin{Bmatrix} x > 0 \\ x < 0 \end{Bmatrix}.$$

In terms of the average value of the scattering amplitude defined in (7), i.e., in terms of $\bar{G}\,(\phi, -\alpha) = \sum \bar{B}_n\, e^{in\phi}$, we rewrite (22) as

$$\bar{\psi} = (2\rho/k\cos\alpha)e^{ikr\cos(\theta+\alpha)}\,\bar{G}\,(-\alpha, -\alpha)\ ,\quad x > 0\ ;$$

(23)

$$\bar{\psi} = (2\rho/k\cos\alpha)e^{-ikr\cos(\theta-\alpha)}\,\bar{G}(\pi + \alpha, -\alpha)\ ,\quad x < 0.$$

The average scattered wave thus consists of a transmitted and a reflected plane wave; the results therefore correspond to those obtained for a plane interface.

Were we to use the single-scattered value of $\bar{G}$, i.e., C as in (7), then $\bar{\psi}$ of (23) would approach infinity as α approaches grazing incidence $(\pi/2)$. This is physically meaningless and indicates the failure of the single-

scattering approximation. One of the principal results of this paper is to show that the multiple-scattered value of $\bar{\psi}$ approaches the negative of the incident wave as grazing incidence is approached; i. e., in the limit the results reduce to those for a dielectric plane.

We note that (23) can also be obtained by replacing $H_n(kr')$ in (20) by its asymptotic value $i^{-n} H(r')$ as in (6) and evaluating the integral by the method of stationary phase; this was done for a slightly simpler case in a previous report[13]. This second procedure also yields (23) for cylinders of arbitrary cross-section provided that $\bar{G}(\theta', -\alpha)$ is a slowly varying function of y' compared to $\exp(ikr' - iky' \sin\alpha)$.

In the following three Sections we consider a closed-form approximation for the scattering coefficients $\bar{B}_n$: in Section 3.1 we derive it; in Section 3.2 we analyze it; in Section 3.3 we discuss its range of validity.

3.1 Derivation of a Closed-form Approximation for the Scattering Coefficients

In order to obtain (18) in closed form we introduce the approximation

(24) $$\langle B_{ms'} \rangle_{ss'} \approx \langle B_{ms'} \rangle_{s'} = \bar{B}_m ;$$

i.e., we assume that the average scattering coefficient when two variables are held fixed is approximately equal to the corresponding value when one variable is held fixed. In Section 3.3 we interpret this approximation from the point of view of successive scattering and discuss its range of validity. (In references 4 and 5, this approximation is used tacitly. Foldy[6] (p. 111) and Lax[7] (p. 624) were the first to use it explicitly.)

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13. V. Twersky, Research Report EM-22, Mathematics Research Group, Washington Square College, NYU, 1951, p. 15 f .

In view of (24) we may now replace $(y_s - y_{s'})$ in (18) by u and write

$$(25)\quad \bar{B}_n = A_n(e^{in\alpha} + f_n^- + f_n^+)\ ,\quad f_n^{\pm} = i^{\mp n} \sum_m \bar{B}_m i^{\mp m}\left[i^{n-m}\int_0^{\infty} e^{\mp iku\sin\alpha} H_{n-m}(ku)p(u)du\right],$$

where we replaced $\theta_{s's}$ in (18) by its values $\mp \pi/2$ for $s' \gtrless s$, and introduced $p(u) = \sum_{\sigma=1}^{\infty} p_\sigma(u)$. Introducing the approximation of (6), i.e., $H_{n-m}(ku) = i^{-n+m} H(u)$, we may replace the function in square brackets in (25) by

$$(26)\qquad \bar{U}_{\pm} = \int_0^{\infty} e^{\mp iku\sin\alpha} H p(u)\, du.$$

Consequently (25) reduces to

$$(27)\quad \bar{B}_n = A_n(e^{in\alpha} + i^n \bar{U}_- \sum \bar{B}_m i^{-m} + i^{-n} \bar{U}_+ \sum \bar{B}_m i^m)\ .$$

Multiplying $\bar{B}_n$ by $\sum_n e^{in\phi}$ and rewriting the results in terms of

$$\bar{G}(\phi, -\alpha) = \sum \bar{B}_n e^{in\phi}\ ,\ C(\phi, -\alpha) = \sum A_n e^{in(\phi+\alpha)}\ ,$$

we obtain

$$(28)\quad \bar{G}(\phi, -\alpha) = C(\phi, -\alpha) + C(\phi, -\pi/2)\bar{U}_-\bar{G}(-\pi/2, -\alpha) + C(\phi, \pi/2)\bar{U}_+\bar{G}(\pi/2, -\alpha)\ .$$

We can now determine $\bar{G}(\phi, -\alpha)$ explicitly in terms of the known functions C and $\bar{U}_{\pm}$.

We substitute $\phi = \pm \pi/2$ into $\bar{G}(\phi, -\alpha)$ and obtain two simultaneous equations, or the matrix equation

$$(29) \qquad \begin{bmatrix} C_o \bar{U}_- - 1 & C_\pi \bar{U}_+ \\ C_\pi \bar{U}_- & C_o \bar{U}_+ - 1 \end{bmatrix} \begin{bmatrix} \bar{G}(-\pi/2, -\alpha) \\ \bar{G}(\pi/2, -\alpha) \end{bmatrix} = - \begin{bmatrix} C(-\pi/2, -\alpha) \\ C(\pi/2, -\alpha) \end{bmatrix} ,$$

$$C_o \equiv C(\pi/2, \pi/2) \quad , \quad C_\pi \equiv C(-\pi/2, \pi/2),$$

where C_o and C_π are the forward and back scattered amplitudes for the isolated cylinder. Solving this equation for the unknown G_s' , we find

$$(30) \qquad \bar{G}(\pm\pi/2, -\alpha) = \frac{(1 - \bar{U}_- C_o)\, C(\pm\pi/2, -\alpha) + \bar{U}_- C_\pi C(\mp\pi/2, -\alpha)}{1 - C_o(\bar{U}_+ + \bar{U}_-) - \bar{U}_+ \bar{U}_- (C_\pi^2 - C_o^2)} ,$$

which is formally identical with the results obtained for the grating[3,9] and has essentially the same interpretation. (The grating is a degenerate case of the distribution we treat.) Substituting (30) into (28) yields the required closed-form representation for the average scattering amplitude.

Since we made use of approximation (6) in order to reduce (25) to (27), our result is restriced to the range $u_{min} = b \gg k^{-1}$, $b \gg a$ - i.e., each cylinder must be in the far fields of its neighbors. However, we can show, at least for certain cases, that the formalism of (27) and consequently the final results (28) and (30) are valid for arbitrary b. Eq. (27) holds for all b provided that the function in square brackets in (25) is independent of n and m, i.e., can be written in the form $\bar{U}_{\pm}$

(or, more generally, in the form $F_{\pm}\, e^{\pm in\beta \pm im\gamma}$, where β and γ may be complex). We mention two special cases where this requirement is fulfilled.

If the average separation of neighbors is large compared to their minimum separation, i.e., if $\bar{b} \gg b$, then from (15) we may write $p(u) = 0$ for $u < b$ and $p(u) = \rho$ for $u > b$. Assuming $\bar{b} \gg b$ and also that $\alpha = 0$, we may write for $[\ ]$ in (25):

$$\rho i^{n-m} \int_b^\infty H_{n-m}(ku)du = i^{n-m}\, \rho b \int_o^\infty H_{n-m}(kb[\,1 + v\,])\, dv \ .$$

We now replace $1 + v$ by $\sqrt{1 + v^2}$ (this introduces negligible error) and make use of (21) with $q = kb$ and $p = 0$ to obtain

$$i^{m-n}\ \frac{\rho b}{2} \int_{-\infty}^{\infty} H_{n-m}\left(kb\ \sqrt{1 + v^2}\right) dv \ = \ \frac{\rho}{k}\ e^{ikb} \ .$$

Consequently we have

$$(31) \qquad \bar{U}_{\pm}\,(\rho) \ \cong \ \frac{\rho}{k}\, e^{ikb} \ , \ \bar{b} \gg b \ , \ \alpha = 0$$

which reduces to ρ/k for a uniform distribution.

Another special case for which the function in square brackets in (25) is independent of n and m for arbitrary values of b is the case of isotropic scatterers, i.e., the case where $C(\phi, -\alpha)$ reduces to A_o as in (5); here we have

$$(32) \qquad \bar{U}_{\pm} = \int_o^\infty H_o(ku) e^{\pm iku\sin\alpha}\, p(u)du \ , \ C = A_o \ , \ \bar{G} \ = \ \frac{A_o}{1 - A_o(\bar{U}_+ + \bar{U}_-)} \ .$$

For the uniform distribution we evaluate $\bar{U}_{\pm}$ exactly [14] to obtain

$$(32') \quad \bar{U}_{\pm} = \rho \int_0^{\infty} H_0(ku) e^{\mp iku \sin\alpha}\, du = \frac{\rho}{k \cos\alpha}\left(1 \mp \frac{2\alpha}{\pi}\right).$$

(Note that if $\alpha = 0$, then (32') reduces to (31) with $b = 0$.)

A principal result can be deduced immediately from (32') and (23): as $\alpha \to \pi/2$ in (32') we have $\bar{U}_+ \to 2\rho/\pi k$ and $\bar{U}_- \to 2\rho/k \cos\alpha$; consequently $\bar{G} = A_0 / \left[1 - A_0(\bar{U}_+ + \bar{U}_-)\right] \to -1/U_- = -k\cos\alpha/2\rho$. Substituting this value of $\bar{G}$ into (23), we find $\bar{\psi} \to -e^{-iky} = -\psi_i$ as $\alpha \to \pi/2$. We discuss this further for a more general case in the following section.

From (31) and (32) we see that the closed-form representation for $\bar{G}$ is not necessarily restricted to $kb \gg 1$, $b \gg a$; for simplicity, however, we restrict ourselves to this range in what follows. A more rigorous representation for $\bar{G}(\phi, -\alpha)$ for the case where only $\bar{B}_0$ and $\bar{B}_{\pm 1}$ are retained is given in the Appendix.

3.2 Analysis of the Functions Involved

The effects of multiple scattering are determined largely by the behavior of the functions $\bar{U}_{\pm}$ of (26). If we write $p(u) = \rho + [p(u)-\rho]$, where $p(u)$ is given by eqs. (12) to (16), then we may write

$$(33) \quad \bar{U}_{\pm}(p) = \sqrt{\frac{2}{i\pi}} \int_0^{\infty} \frac{e^{iku(1 \pm \sin\alpha)}}{\sqrt{ku}}\, p(u)du \equiv \bar{U}_{\pm}(\rho) + \bar{U}_{\pm}(p - \rho), \quad u > b,$$

where $\bar{U}_{\pm}(p - \rho) \to 0$ if $(\bar{b} - b)/b \to \infty$. We first consider $\bar{U}_{\pm}(\rho)$.

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14. W. Magnus and F. Oberhettinger, Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik (Springer, Berlin, 1948) p. 180, line 2.

The functions $\bar{U}_{\pm}(\rho)$ may be expressed in terms of the tabulated Fresnel integrals[15] as

$$(34)\quad \bar{U}_{\pm}(\rho) = \frac{2\rho}{k\sqrt{(1\pm\sin\alpha)i}}\left\{F\left(kd\left[1\pm\sin\alpha\right]\right)-F\left(kb\left[1\pm\sin\alpha\right]\right)\right\} = \bar{U}_{\pm}(d) - \bar{U}_{\pm}(b)\ ,\quad d\to\infty,$$

where we use a finite upper limit d to facilitate discussion (and also so that the expressions may be used heuristically for a finite range). The first term of (34) is the result for a uniform distribution:

$$(35)\qquad \bar{U}_{\pm}(d) = \frac{\rho}{k}\sqrt{\frac{2}{1\pm\sin\alpha}}\ .$$

Hence except for normal incidence, $\bar{U}_-$ is greater than $\bar{U}_+$ and their difference increases as α increases, e.g., $\bar{U}_-/\bar{U}_+ \gg 11$ for $\alpha > 80^\circ$. If $\alpha \to \pi/2$, then

$$(36)\quad \bar{U}_-(d) \to \frac{2\rho}{k\cos\alpha}\ ,\quad \bar{U}_+(d) \to \frac{\sqrt{2}\,\rho}{k}\ ,\quad \alpha\to\pi/2\ .$$

(Note that the results of (35) and (36) differ negligibly from those of (32') obtained for arbitrary b.)

The second term of (34), $-\bar{U}_{\pm}(b)$, is the contribution of the 'hole' of width 2b (2b defines the range within which only one scatterer is found). For large values of $kb(1 \pm \sin\alpha)$, the largest $\bar{U}_{\pm}(b)$ cancels $\bar{U}_{\pm}(d)$ and hence $\bar{U}_{\pm}(\rho)$ is small:

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15. The present function F(u) equals C(u) + iS(u) in terms of the functions tabulated in E. Jahnke and F. Emde, Tables of Functions (Dover, NY, 1945) p. 35. If $u \gg 1$, then $F(u)\sim\sqrt{i/2}\left[1-e^{iu}\sqrt{i/\pi u}(1-i/2u\ldots)\right]$; if $u < 1$, then $F(u) = \sqrt{2u/\pi}\,(1 - iu/3 - u^2/10\ldots\ldots)$.

$$(37)\quad \bar{U}_{\pm}(\rho) = \bar{U}_{\pm}(d) - \bar{U}_{\pm}(b) \sim \frac{\rho}{k(1\pm\sin\alpha)}\sqrt{\frac{2i}{\pi kb}}\, e^{ikb(1\pm\sin\alpha)}, \quad kb(1\pm\sin\alpha) \gg 1 \ .$$

Eq. (37) holds for $\bar{U}_+$ for all α, and holds also for $\bar{U}_-$ provided that α is not large; however, if $\alpha \to \pi/2$, then

$$(38)\qquad \bar{U}_-(b) \to \frac{2\rho}{k}\sqrt{\frac{2kb}{i\pi}} \ , \quad \alpha \to \pi/2 \ ,$$

which is negligible compared to $\bar{U}_-(d)$ of (35). Consequently, since the various C's in (30) are not generally large, we may neglect $\bar{U}_+(\rho)$ for all α, and also neglect $\bar{U}_-(\rho)$ except for large values of α; for $\alpha \to \pi/2$ we have $\bar{U}_-(\rho) \to \bar{U}_-(d) \to 2\rho/k\cos\alpha$.

Hence for the 'more random' distributions (i.e., for $\bar{b} \gg b$) we may approximate $G(\phi, -\alpha)$ as in (28) by its single-scattered value $C(\phi, -\alpha)$ unless α is very large; furthermore when we do take multiple scattering into account, $\bar{U}_+$ is still negligible compared to $\bar{U}_-$ and we may use

$$(39)\quad \bar{G}(\phi, -\alpha) \approx C(\phi, -\alpha) + \frac{C(\phi, -\pi/2)\, C(-\pi/2, -\alpha)\, \bar{U}_-}{1 - C_o\bar{U}_-} \ .$$

For the vicinity of grazing incidence (39) reduces to

$$(40)\quad \bar{G}(\phi,-\alpha) \to \frac{C(\phi,-\alpha)}{1-C_o\bar{U}_-} \ , \quad \bar{U}_- \to \frac{2\rho}{k\cos\alpha} \ , \quad \alpha \to \frac{\pi}{2} \ .$$

Consequently $\bar{\psi}$ of (23) approaches

$$(41)\quad \bar{\psi} \to -\frac{2\rho}{k\cos\alpha}\frac{e^{-iky}}{\bar{U}_-} \to -e^{-iky} = -\psi_i \ , \quad \alpha \to \pi/2 \ .$$

Hence, whereas the single-scattered value of $\bar{\psi}$ approaches ∞ as $\alpha \to \pi/2$, the multiple-scattered value approaches the negative of the incident wave. This result (which we believe has not previously been derived analytically) is significant in 'predicting' that a striated surface should behave like a perfect reflector near grazing incidence[16]; we discuss this further in Section 6.

In similar problems multiple scattering has been ignored on the grounds that 'no coherent phase relations could exist because of the randomness of the distribution'. The following naïve considerations show that this remark is more or less valid for a uniform distribution except near grazing incidence. For $p = \rho$ we may regard the integral of (33) as the 'sum' of the excitations from all scatterers rather than as the average over all configurations of the excitation arising from one element. From this viewpoint, the phase of each elementary excitation consists of two terms: the term in ku arises from the distance the excitation travels, while that in $\pm$ iku sinα is the relative phase difference introduced by the incident wave. In general, the elementary excitations are out of phase, since each corresponds to a different value of u, and their resultant is a relatively small quantity. For $\alpha \to \pi/2$, however, the excitations in $\bar{U}_-$ are all in phase, the distance factor canceling that arising from the incident plane wave; hence, in spite of the 'randomness' of the distribution, the resultant of all the excitations is a large quantity which must be taken into account. We note that since the phase is inversely proportional to λ, the resultant at small values of α is larger for the longer wavelengths; from this we deduce that the longer wavelengths are the first to be affected by multiple scattering as $\alpha \to \pi/2$. We also note that at the limit $\alpha = \pi/2$, the above considerations are independent of the probability aspects of the problem and apply for a single configuration; that is, in any configuration

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16. Note that at $\alpha = \pi/2$, $\psi_i + \bar{\psi}$ vanishes; this merely indicates that a solution to the problem does not exist in terms of simple plane waves. Here we must seek solutions in terms of 'surface waves' which are exponentially damped normal to the surface. See P.M. Morse, Vibration and Sound (McGraw-Hill, NY, 1948) pp. 368, 371; J.A. Stratton, Electromagnetic Theory (McGraw-Hill, NY, 1941) p. 516 ff.

the excitations reaching each cylinder from its neighbor lying between the cylinder and the source are all in phase for $\alpha = \pi/2$. From this we infer that the variance of ψ (i.e., the mean square deviation) approaches zero for $\alpha \to \pi/2$; we discuss this further in Section 4.

We now consider a numerical example and discuss the wavelength dependence of $\bar{\bar{U}}_{\pm}(\rho)$ of (34); we seek a criterion in terms of kb and α for neglecting multiple scattering. In Figs. 2a and b the real and imaginary parts of $\bar{\bar{U}}_{+}(\rho)(k/\rho)\sqrt{i} \equiv R + iI$ are plotted as functions of α for kb = 10; the positive and negative values of α correspond to $\bar{\bar{U}}_{+}$ and $\bar{\bar{U}}_{-}$ of (34) respectively. It can be seen that $\bar{\bar{U}}_{+}$ is in general negligible compared to $\bar{\bar{U}}_{-}$ and that $\bar{\bar{U}}_{-}$ is large only for $\alpha > 80^{\circ}$ (i.e., when $\bar{\bar{U}}_{-}(\rho) \approx \bar{\bar{U}}_{-}(d)$). Thus as α increases, the excitations traveling along the forward projection of the incident wave normal on the plane of the configuration (along -y in Fig. 1) become much larger than those traveling backward as if an 'obliquity factor' were in operation.

The lower limit of the angular range in which multiple scattering is significiant is determined primarily by the value of kb: (37) may be used up to larger values of α for the larger values of kb, i.e., multiple scattering may be neglected over a larger range of α for the smaller wavelengths. We may take $kb(1-\sin\alpha) = 0.1$ as a convenient relation to determine the lower limit of α, say α_c; this corresponds to $|F(b)| < 0.02|F(d)|$. Although the multiple scattering effects are highly selective for all α they may be ignored for $\alpha < \alpha_c$. Hence for $\alpha < \alpha_c$ we use $G(\phi,-\alpha) \approx C(\phi,-\alpha)$, and for $\alpha > \alpha_c$ we use $G(\phi,-\alpha)$ as in (39).

Up to this point we have approximated $p(u)$ by its step function limit (i.e., $p(u) = 0$ for $u < b$ and $p(u) = \rho$ for $u > b$), and considered only $\bar{\bar{U}}_{\pm}(\rho)$; we now investigate the behavior of the terms $\bar{\bar{U}}_{\pm}(p - \rho)$ of (33). We show that the above remarks in regard to $\alpha \to \pi/2$ are not vitiated, and also that multiple scattering is significant for certain values of kb and $\alpha \neq \pi/2$ if there exists a high degree of order in the distribution of neighbors (i.e., if (16) holds).

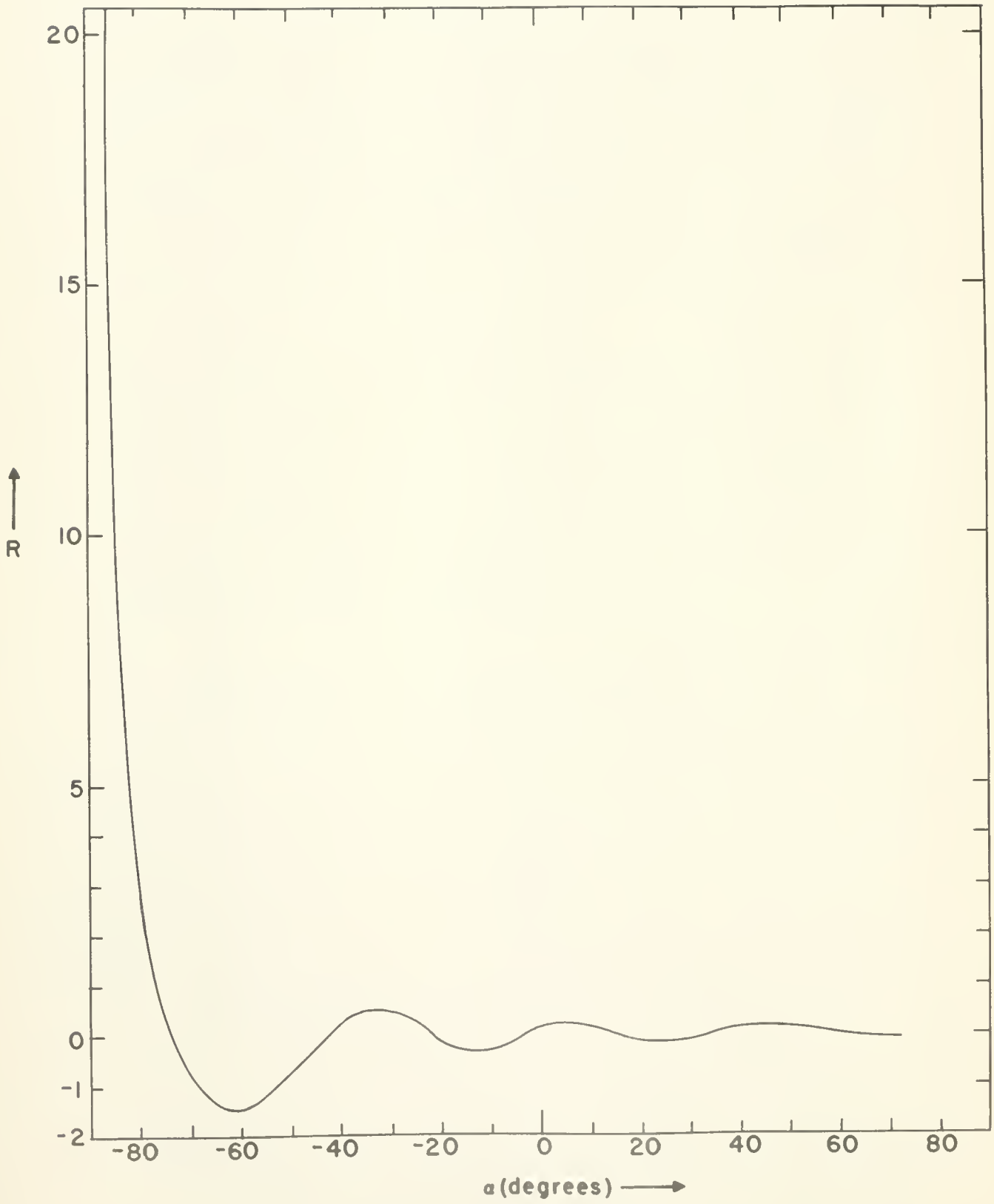

Figure 2a

Figure 2b

Replacing $p(u)$ in the integral of (33) by $p(u) - \rho$ as obtained from (14), we find

$$(42)\quad \bar{U}_{\pm}(p-\rho) = 2\sum_{|\nu|=1}^{\infty} \frac{F(dW_\nu) - F(bW_\nu)}{\left[b+\delta\exp(bw_\nu)\right]\sqrt{ikW_\nu}}\ ,\quad W_\nu = k(1 \pm \sin\alpha) - iw_\nu\ .$$

For large values of W_ν we obtain

$$(42')\quad \bar{U}_{\pm}(p-\rho) \sim \sqrt{\frac{2i}{\pi kb}}\, e^{ikb(1\pm\sin\alpha)} \sum_{|\nu|=1}^{\infty} \frac{\exp(bw_\nu)}{\left[k(1\pm\sin\alpha)-iw_\nu\right]\left[b+\delta\exp(bw_\nu)\right]}\ ,$$

the term $\nu = 0$ being given by (37). The effects of these terms on our previous results depend largely on $\delta/b = (\bar{b} - b)/b$ and must in general be determined numerically. For $\alpha \to \pi/2$, however, (42') remains finite and is therefore negligible compared to $\bar{U}_{-}(d)$; consequently (41) still holds for $\alpha \to \pi/2$, regardless of the value of δ/b (i.e., of the degree of order of the distribution).

We may use (42') instead of (42) provided that $W_\nu \neq 0$. However, if δ/b is small, then $w_\nu \approx i2\nu\pi\rho$ (see (16)), and consequently $W_\nu = \rho\left[k\bar{b}\ (1 \pm \sin\alpha) + 2\nu\pi\right]$ vanishes for negative values of ν for certain values of kb and α; i.e., $W_\nu \to 0$ if $k\bar{b}(1\pm\sin\alpha) \to -2\nu\pi$, $\nu = -|\nu|$. These values determine the 'Rayleigh wavelengths' corresponding to the 'Wood anomalies' which we discussed in detail in ref. 3. For these values we neglect $F(bW_\nu)$ in (42) and approximate $\bar{U}_{\pm}(p)$ by the term $\nu = -n$ to obtain

$$(43)\quad \bar{U}_{\pm}(p) \approx \frac{2}{\sqrt{i\rho k\left[k\bar{b}(1\pm\sin\alpha)-2n\pi\right]}}\ \frac{F\left\{\rho d\left[kb(1\pm\sin\alpha)-an\pi\right]\right\}}{b+\delta e^{-i2n\pi\rho b}}\ .$$

This expression behaves essentially like the result obtained for the grating[3,9]. If we use a finite value of d and impose the restriction $|C_o \bar{U}_{\pm}| < 1$, then our discussion in ref. 3 applies - these restrictions, used heuristically in ref. 3, yielded results for the widths of the 'anomalies', their resolution, etc., which agreed satisfactorily with the experimental data. If we do not impose these restrictions, then Karp's treatment[9] is appropriate.

Letting d become infinite in (43) and neglecting small quantities we obtain

$$\bar{U}_{\pm}(p) \approx \sqrt{\frac{2\rho}{k\left[k\bar{b}(1 \pm \sin\alpha) - 2n\bar{\pi}\right]}} \quad ; \tag{43'}$$

(43') reduces to $\bar{U}_{\pm}(d)$ as in (35) for n = 0 and hence behaves for $\alpha \to \pi/2$ as discussed previously. If (43') holds for $\bar{U}_-$ while (37) and (42') hold for $\bar{U}_+$, then for $kb(1 - \sin\alpha) \to 2n\pi$ the scattering amplitude reduces to (39). Proceeding to the limit (for finite values of the single-scattered wave) we obtain

$$\bar{G}(\phi, -\alpha) \to C(\phi, -\alpha) - C(\phi, -\pi/2)\, C(-\pi/2, -\alpha)/C_o \quad ; \tag{39'}$$

whether the amplitude is larger or smaller than its single-scattered value depends on the physical constants of the cylinders. (Note that $G \to 0$ for isotropic scatterers.) Similarly if $\bar{U}_+ \to \infty$ while $\bar{U}_-$ remains constant, we merely replace $\pm \pi/2$ in (39) by $\mp \pi/2$. If (43') holds for both $\bar{U}_+$ and $\bar{U}_-$, i.e., if $kb(1 \pm \sin\alpha) \to 2n_{\pm}\pi$, then we let $\bar{U}_+$ and $\bar{U}_-$ both approach ∞ in (30) to obtain the appropriate result.

The expressions we have derived for $\bar{U}_{\pm}(p)$ are most useful for limiting values of δ/b. For arbitrary δ/b we must employ (33) with p(u) as given by (11) and (12). To facilitate numerical

analysis, we derive a more useful representation by proceeding essentially as for (13). We introduce the new variable $v = (u - \sigma b)/\delta$ into (33) and write

$$(44)\quad \bar{U}_{\pm}(p) = \sqrt{\frac{2}{i\pi k}} \sum_{\sigma} \frac{e^{i\sigma b\gamma}}{(\sigma-1)!} \int_0^{\infty} \frac{e^{-v(1-i\gamma\delta)}}{\sqrt{\delta v + \sigma b}}\, v^{\sigma-1} dv, \quad \gamma = k(1 \pm \sin\alpha) \ .$$

Making use of $2\int_0^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}$, $a = \delta v + \sigma b$, we evaluate the resulting integral over v; summing over v and introducing the new variable $w = x^2 - i\gamma b$ we obtain

$$(45)\quad \bar{U}_{\pm}(p) = \frac{1}{\pi}\sqrt{\frac{2}{ik}} \int_{o\ -i}^{\infty\ -i} \frac{h(w)}{\sqrt{w + i\gamma}}\, dw,$$

where h(w) is the Laplace transform of p(u) as in (13). We note that we could now proceed as for (14) and express (35) as a sum of residues plus the integral around the branch cut; initiating this procedure we see that the pole at the origin yields $\bar{U}_{\pm}$ (d) as in (35).

3.3 The Range of Validity of the Closed-form Representation

In this section we discuss the significance and the range of validity of eq. (24), the approximation on which our closed-form representation of $\bar{G}(\phi, \alpha)$ hinges. It is clear that since we followed a procedure that would be rigorous for a periodic configuration, our results are valid at least for $\delta/b \longrightarrow 0$; however, we have as yet no physical picture of what the approximation entails. To investigate the approximation we revert to the point of view of successive scattering.

We restrict discussion initially to the case of isotropic scatterers;here we require the average of (5) with one variable held fixed. For brevity we rewrite (5) as

$$B_{os} = A_o\Big(1 + \sum_{s'}{}' U_{ss'} B_{os'}\Big) \ , \quad U_{ss'} = H_o(k|y_s - y_{s'}|)\exp\left[ik(y_s - y_{s'})\sin\alpha\right] \ .$$

On solving for B_{os} by successive substitutions for $B_{os'}$ we obtain the series

$$B_{os} = A_o(1 + A_o \sum_{s'}{}' U_{ss'} + A_o^2 \sum_{s'}{}' \sum_{s''}{}' U_{ss'}U_{s's''} + \ldots.) \ , \quad s' \neq s, \quad s'' \neq s' \ ,$$

where the first three 'orders of scattering' have been written explicitly. We obtain $< B_{os}>_s$ by multiplying the above by $P(y_s|y_1 \ldots . y_{s-1}, y_{s+1} \ldots . y_N)$ and integrating with respect to all variables $y_{s'} \neq y_s$. Thus on integrating out the extraneous variables, we find

$$< B_{os}>_s = A_o + A_o^2 \sum_{s'}{}' \int U_{ss'} P(y_s|y_{s'}) dy_{s'} + A_o^3 \sum_{s'}{}' \sum_{s''}{}' \iint U_{ss'}U_{s's''} P(y_s|y_{s'}, y_{s''}) dy_{s'} dy_{s''} + \ldots$$

(46)

$$\equiv A_o + A_o^2 \sum_{s'}{}' < U_{ss'} > + A_o^3 \sum_{s'}{}' \sum_{s''}{}' < U_{ss'}U_{s's''} > + \ldots . \quad .$$

(This same result could of course have been obtained by simplifying (18), expressing $< B_{os}>_s$, and then expanding by integration.) For the ensemble defined in Section 2.2 we have $P(y_s|y_{s'}, y_{s''}) = P(y_s|y_{s'})P(y_{s'}|y_{s''})$, etc.; consequently certain terms of the third order of scattering correspond to the square of terms in the second order, etc.

The single sum contains N - 1 terms of essentially the same form. The double sum contains (N - 1)(N - 2) terms like $< U_{ss'} >^2$ and N - 1 terms like $< U_{ss'}U_{s's} > = \int P(y_s|y_{s'}) \ U_{ss'} \ U_{s's} \ dy_{s'}$

$= \int P(y_s|y_{s'}) \ H_o^2(k|y_s - y_{s'}|) \ dy_{s'}$; the two kinds arise since the requirement $s \neq s' \neq s''$ does not rule out $s'' = s$; i.e., the third order of scattering of cylinder s may arise from its own first order rescattered by s'. We may rewrite the double sum as $< \sum{}' U_{ss'} >^2$

$+ \sum{}' \left[< U_{ss'}U_{s's} > - < U_{ss'} >< U_{s's} > \right]$, where the first term equals $(\bar{U}_+ + \bar{U}_-)^2$ in terms of the functions of the preceding Section. Similarly the multiple sums of the higher orders of scattering may be written as $(\bar{U}_+ + \bar{U}_-)^n$ plus 'correction terms' which are similar to but much more complicated than the present function in square brackets.

The leading terms of all orders of scattering form a geometric progression and may be summed directly to obtain $\bar{G}$ as in (32). (The above may

be readily extended to non-isotropic scatterers; then we would use the matrix summation method developed by Magnus for the grating[3] in order to obtain $\bar{G}$ as given by (28) and (30).) However, it does not appear that the remaining terms can be summed. It can be seen that ignoring these terms is equivalent to starting the successive scattering process with the average single-scattered wave; this implies that each stage of scattering (each sum) is averaged separately - as if each scatterer 'recoiled' on scattering and thus prevented the occurrence of mutual path differences in subsequent stages of scattering (i.e., common terms in the various multiple sums). Thus ignoring the distinct character of the terms which arise from the presence of identical path differences in subsequent stages of scattering is equivalent to replacing the average scattering coefficient with two variables held fixed by the average coefficient for one variable held fixed.

We may get some idea of the range of validity of our approximation by considering the function $\sum{}' \left[\langle U_{ss'} U_{s's} \rangle - \langle U_{ss'} \rangle \langle U_{s's} \rangle \right]$ and determining for which values of the parameters it may be neglected. Thus if $\delta/b \to 0$, we find that each term inside the square brackets approaches $H_o^2 (kb|s - s'|)$ (the limiting form of $\langle U_{ss'} \rangle$ is obtained from (44)). Hence, as expected, the approximation is valid for the more ordered distribution. On the other hand if δ/b is large, then we may neglect the term $\sum{}' \langle U_{ss'} \rangle \langle U_{s's} \rangle = \bar{U}_+ \bar{U}_- / (N - 1)$ and use

$$(47) \qquad \sum{}' \langle U_{ss'} U_{s's} \rangle = \frac{4\rho}{i\pi k} \int_b^\infty \frac{e^{i2ku}}{u} \, du = \frac{i2\rho}{\pi k} \mathrm{Ei}(i2kb) \quad ,$$

where $\mathrm{Ei}(iu) = \mathrm{Ci} + i\,\mathrm{si}(u)$ in terms of the cosine and sine integrals of ref. 15, pp. 3-9. Although (47) is in general larger than $\bar{U}_+ \bar{U}_- / (N - 1)$ (as was expected physically from the 'resonance' effect involved), it may still be ignored for $\alpha \to \pi/2$ since $(\bar{U}_+ + \bar{U}_-)^2 \propto \sec^2 \alpha$ is by far the dominant term.

Essentially the same considerations apply for the higher orders of scattering; there instead of (47), we deal with functions of the order of $\left[2H(kb)\right]^n i\rho/nk$.

Hence we may regard our closed-form representation as valid either when there is a high degree of order in the distribution of neighbors, or near grazing incidence for any distribution. We are thus justified in using (30) for the very range in which we would be inclined to use it, i.e., the range where (30) indicates that the effect of multiple scattering may be large. Thus our approximation is valid at least where $\bar{U}_{\pm}$ (essentially the single-scattered wave evaluated in the plane of the axes) is very large.

If the effects of multiple scattering as indicated by $\bar{U}_{\pm}$ are small, then we may follow one of at least two alternative procedures:

1) We may use only the first two orders of scattering as obtained from (4) by iteration; here the averages involve only $W_1(y_s)$ and $W_2(y_s, y_{s'})$ and may be performed explicitly even when the end effects for a finite range are taken into account. This procedure is appropriate for $ka \to 0$.

2) We may retain all orders of scattering in the iterated form of (4), average formally, and then expand each order in terms of a suitable small parameter; this differs from 1) in that some portion of each order of scattering is retained. Thus for the more random distributions we expand in terms of ρ/k. We now illustrate this procedure for $\alpha = 0$ for the case $kb \gg 1$.

Retaining only terms to the order of ρ/k in $\langle B_{os} \rangle_s = \bar{B}_o$ of (46) we write

$$\bar{B}_o \approx A_o\left[1+A_o \sum_{s'}{}' \langle U_{ss'} \rangle + A_o^2 \sum{}' \langle U_{ss'} U_{s's} \rangle + A_o^3 \sum{}' \langle U_{ss'} U_{s's} U_{ss'} \rangle + \ldots.\right] \tag{48}$$

$$= A_o\left[1+\rho(A_o \overline{D} + A_o^2 \overline{D^2} + \ldots.)\right] = A_o\left[1+\rho \sum_{n=1}^{\infty} A_o^n \overline{D^n}\right] ,$$

where $\overline{D^n} = 2\int_b^{\infty} \left[H(ku)\right]^n du \sim \dfrac{\left[H(kb)\right]^n 2i}{nk}$, $kb \gg 1$.

Hence

$$(48') \quad \bar{B}_o \approx A_o \left\{ 1 - \frac{i2\rho}{k} \log\left[1 - A_o H(kb)\right] \right\} , \quad H(kb) = \sqrt{\frac{2}{i\pi kb}}\, e^{ikb}$$

(compare with (32) and (32')) which were obtained by means of approximation (24)). $\bar{B}_o$ of (48') is the average scattering coefficient correct to the power of ρ/k; since the ρ/k term of $\bar{B}_o$ corresponds to the $(\rho/k)^2$ term of $\bar{\psi}$, we see that the single-scattered value is correct to lowest order in ρ/k.

If $\alpha \neq 0$, then the odd powers of $\overline{D^n}$ in (48) are to be replaced by

$$2\int_b^\infty \left[H(ku)\right]^n \cos(kusin\alpha)du \sim \frac{i}{k}\left[H(kb)\right]^n \left[\frac{e^{ikusin\alpha}}{n + sin\alpha} + \frac{e^{-ikusin\alpha}}{n - sin\alpha}\right] .$$

It can be seen that for $\alpha \to \pi/2$, the only term in ρ/k which is significant is the term $n = 1$. Hence for $\alpha \to \pi/2$ we are to employ (32') which is essentially an expansion in powers of $\rho/k \cos\alpha$.

4. The Average Intensity

In this section we derive the corresponding closed-form representation for the ensemble average of the multiple-scattered intensity $|\psi|^2$. For simplicity we work with the approximate form of ψ as in (8) rather than with the rigorous representation as in (4).

The scattered intensity for a particular configuration may be written as

$$(49) \quad |\psi|^2 = \sum_s \sum_t \psi_s \psi_t^* = \sum_s \sum_t \exp\left[-ik(y_s - y_t)sin\alpha\right] H(r_s) H^*(r_t) G_s(\varrho_s, -\alpha) G_t^*(\varrho_t, -\alpha)$$

where $G_s G_t^*$ is given by (8). The average of $|\psi|^2$ over the ensemble of configurations defined in Section 2.2 may be written as

$$\langle|\psi|^2\rangle = \sum_s \sum_t \langle \psi_s \psi_t^* \rangle = \sum_s \sum_t{}' \langle \psi_s \psi_t^* \rangle + \sum_s \langle|\psi_s|^2\rangle$$

$$(50) \quad = \sum_s \sum_t{}' \iint W_2(y_s, y_t) \exp\left[-ik(y_s - y_t)\sin\alpha\right] H(r_s) H^*(r_t) \langle G_s(\theta_s, -\alpha) G_t^*(\theta_t, -\alpha)\rangle_{st} dy_s dy_t$$

$$+ \sum_s \int W_1(y_s) |H(r_s)|^2 \langle |G_s(\theta_s, -\alpha)|^2 \rangle_s \, dy_s \quad ,$$

where formal expressions for $\langle G_s G_t^* \rangle_{st}$ and $\langle|G_s|^2\rangle_s$ may be obtained from G_s of (8). We omit these for brevity since we will approximate both $\langle G_s G_t^* \rangle$ and $\langle|G_s|^2\rangle_s$ by either $\bar{G}(\theta_s, -\alpha)\bar{G}^*(\theta_t, -\alpha)$, or else simply by $C(\theta_s, -\alpha)C^*(\theta_t, -\alpha)$ if $\bar{G}$ of (28) indicates that the effects of multiple scattering are small. Our procedure is based on the following considerations.

In general, if we ignore the difference between averages with different numbers of variables held fixed, we can reduce the problem of determining $\langle G_s G_t^* \rangle$ to that of solving a set of four simultaneous integral equations. (Unlike $\bar{G}$ or $\langle|G_s|^2\rangle$, $\langle G_s G_t^* \rangle$ is an explicit function of y_s and y_t.) From these equations, or from elementary considerations, we find that $\langle G_s G_t^* \rangle$ differs from $\bar{G}_s \bar{G}_t^*$ essentially in possessing terms of the form

$$(51) \quad \sum_{s'}{}' \left[\langle U_{ss'} U_{ts'}^* \rangle - \bar{U}_{ss'} \bar{U}_{ts'}^*\right] = \sum_{s'}{}' \langle (U_{ss'} - \bar{U}_{ss'})(U_{ts'} - \bar{U}_{ts'})^* \rangle \quad ,$$

$$U_{ss'} = \exp\left[ik(y_s - y_{s'})\sin\alpha\right] H(k|y_s - y_{s'}|) \quad ,$$

where s may equal t. If $s \neq t$, then (51) is the covariance (i.e., the

degree of correlation) of $U_{ss'}$ and $U_{ts'}$; if s = t, then (51) is the variance of $U_{ss'}$. We consider only the case s = t, which is of greater physical interest.

The sum over $s' \lesseqgtr s$ in (51) for the case s = t yields the variance of $\bar{U}_{\pm}$ of (26):

$$(52)\quad V_{\pm} \equiv \sum_{s' \lesseqgtr s} \left[\langle |U_{ss'}|^2 \rangle - |\bar{U}_{ss'}|^2 \right] = \langle |U_{\pm} - \bar{U}_{\pm}|^2 \rangle = \langle |U_{\pm}|^2 \rangle - |\bar{U}_{\pm}|^2, \quad U_{\pm} = \sum_{s' \lesseqgtr s} U_{ss'} .$$

The leading term of (52) may be expressed as

$$(53)\quad \sum_{s' \lesseqgtr s} \langle |U_{ss'}|^2 \rangle = \int_0^\infty p(u) |H(u)|^2 du = \frac{2}{\pi k} \int_0^\infty \frac{p(u)}{u} du = \frac{2}{\pi k} \int_0^\infty h(w)\, dw \quad ,$$

where h(w) is given by (13). The remainder of (52) may be obtained from (44).

It is clear that $V_{\pm}$ must vanish for $\delta/b \to 0$, i.e., for the limiting case of the grating. We find

$$(54)\quad V_{\pm} \sim \frac{2}{\pi k b} \left\{ \log \frac{d}{b} - \log \left[\frac{1 + \gamma^2 \delta^2}{2\delta^2} \right] \right\} , \quad \frac{d}{b} \to \infty , \quad \gamma = k(1 \pm \sin\alpha) \quad .$$

Thus (54) vanishes either for $\delta \to 0$ or $\alpha \to \pi/2$, and becomes infinite as log(d/b) otherwise.

On the other hand for the more random distributions, $\delta/b \gg 1$, we find

$$(55)\quad V_{\pm} \sim \frac{2}{\pi k \delta} \left\{ \log \frac{d}{b} - \frac{1}{\sqrt{1 + \gamma^2 \delta^2}} \left[1 + e \log \left(\frac{\gamma^2 \delta^2}{1 + 2\delta^2} \right) \right] \right\} , \quad \frac{d}{b} \to \infty \quad .$$

If $\alpha \to \pi/2$, then (55) is of the order of $2/\pi$, which is negligible compared to $|\bar{U}_{-}|^2$; for other values of α, (55) approaches infinity as log(d/b).

Hence the variance of $U_{\pm}$ is negligible for $\delta \longrightarrow 0$ or $\alpha \rightarrow \pi/2$ and approaches infinity as $\log (d/b)$, $d \rightarrow \infty$, otherwise*. That $V_{\pm}$ would be small for $\delta \rightarrow 0$ or $\alpha \rightarrow \pi/2$ was of course expected from the discussion of the previous section. However, what we can deduce from the result that $V_{\pm} \longrightarrow \infty$ for other values of δ or α ?

The variance indicates how particular values of a random variable are spread around the mean. If the variance is small, then large deviations from the mean are improbable; if it is large, then not all values are clustered around the mean, and consequently the mean is no longer useful to characterize the behavior of the variable. Our results for $\overline{V}_{\pm}$ thus indicate that $\overline{U}_{\pm}$ is physically significant only when either $\delta \rightarrow 0$ or $\alpha \rightarrow \pi/2$. Our closed-form approximation for $\overline{G}$ is thus valid either when there is a high degree of order in the distribution, or else when grazing incidence is is approached; this agrees with the analysis of Section 3.3. Moreover, since the functions involved in $< G_s G_t^* >$ reduce to those in $|\overline{G}|^2$ when $\delta \rightarrow 0$ or $\alpha \rightarrow \pi/2$, we may then employ $\overline{G}\ \overline{G}^*$ as obtained from (39) rather than $< G_s G_t^* >$; for other values of δ or α we use $C\ C^*$ for both $< G_s G_t^* >$ and $\overline{G}\ \overline{G}^*$.

We show later, in the section on the average energy flux, that the results we obtain by using this approximation are consistant with conservation of energy.

We now return to $<|\psi|^2>$ of (50) and approximate it by

$$<|\psi|^2> = \frac{1}{\pi kd} \sum_s {}' \sum_t \iint P(y_s|y_t) \exp\left\{ik[r_s - r_t - (y_s - y_t)\sin\alpha]\right\} \frac{\overline{G}(\varrho_s, -\alpha)\overline{G}^*(\varrho_t, -\alpha)}{\sqrt{r_s r_t}}\, dy_s dy_t$$

(56)

$$+ \frac{2}{\pi kd} \sum_s \int \frac{|\overline{G}(\varrho_s, -\alpha)|^2}{r_s}\, dy_s \equiv I_2 + I_1$$

where P is given by (10) and (11). Since the function I_1 is essentially the variance of ψ, our results are meaningful only if I_1 is small compared

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* The infinity can be 'removed' by assuming that k is complex or that the distribution is finite; however, these would be different problems.

to $|\psi|^2$. To prevent I_1 from becoming logarithmically infinite, we adopt the following convention: when the effects of multiple scattering as indicated by $\bar{G}$ are small we replace $\bar{G}$ by C and integrate only over a finite range 2d.

We sum over s in I_1 to obtain

$$I_1 = \frac{2\rho}{\pi k} \int \frac{|\bar{G}(\theta', -\alpha)|^2}{r'} \, dy' \tag{57}$$

which is simply the sum of the average intensities scattered by the individual cylinders. For circular cylinders, the integrals in I_1 may be evaluated exactly. In the limit $\alpha \to \pi/2$, we have $|\bar{G}|^2 \to |C(\theta', -\pi/2)|^2 k^2 \cos^2\alpha / |C_o|^2 4\rho^2$; consequently, since the C's remain finite, I_1 vanishes at grazing incidence. On the other hand if $\bar{G} \approx C$, then we integrate only over a finite range in (57); thus for $d \ll r$ we use $\theta' \approx \theta$ and obtain

$$I_1 \approx \frac{2\rho 2d}{\pi k r} |C(\theta, -\alpha)|^2 \equiv N|\psi_o(\theta)|^2 \quad , \tag{58}$$

where N is the number of 'illuminated' scatterers and ψ_o is the wave scattered by a cylinder at the center of the illuminated range. Here I_1 is the 'incoherent' intensity.

If the distribution is uniform, i.e., if $P(y_s|y_t) = (2d)^{-1}$, then I_2 reduces to $|\bar{\psi}|^2$, i.e.,

$$I_2 = |\bar{\psi}|^2 = \left(\frac{2\rho}{k \cos\alpha}\right)^2 \left\{ \begin{array}{l} |\bar{G}(-\alpha,-\alpha)|^2 \\ |\bar{G}(\pi+\alpha,-\alpha)|^2 \end{array} \right\} , \left\{ \begin{array}{l} x > 0 \\ x < 0 \end{array} \right\} , \tag{59}$$

where we replaced (N - 1)/N by unity. If $\alpha \to \pi/2$, then $I_2 \to 1$; see (41).

For the general case we express $P(y_s|y_t)$ in terms of $p_\sigma(u)$ of (11). Here we approximate r_t in the exponent of (56) by using $r_t \approx r_s - (y_t - y_s)\sin\theta_s$; we approximate the slowly varying functions in r_t and θ_t by using $r_t \approx r_s$, $\theta_t \approx \theta_s$. Consequently in terms of $u = |y_s - y_t|$, $\sigma = |s-t|$, we may write

$$I_2 \approx \frac{1}{\pi kd} \sum_{s=-M}^{M} \int_{-\infty}^{\infty} dy_s \, \frac{|\bar{G}(\theta_s,-\alpha)|^2}{r_s} \int_0^{\infty} \left[\sum_{\sigma=1}^{M+s} e^{iu\gamma_s} + \sum_{\sigma=1}^{M-s} e^{-iu\gamma_s} \right] p_\sigma(u)\, du \quad ,$$

(60)

$$\gamma_s = k(\sin\theta_s + \sin\alpha) \quad ,$$

where $2M + 1 = N$; we keep finite limits on the sum to facilitate discussion. The integral over u was evaluated in (13), i.e.,

$$(61) \quad \int_0^{\infty} e^{iu\gamma_s} p_\sigma(u)du = \left[\frac{\exp(ib\gamma_s)}{1-i\delta\gamma_s} \right]^\sigma \equiv \left[Q(\gamma_s) \right]^\sigma .$$

We may now sum over σ and s, and replace the subscript by a prime; we obtain

$$I_2 \approx \frac{1}{\pi kd} \int \frac{|\bar{G}(\theta',-\alpha)|^2}{r'} J(\gamma')d\gamma' \quad ,$$

(62)

$$J(\gamma') = 2\mathrm{Re} \sum_{s=-M}^{M} \sum_{\sigma=1}^{M+s} \left[Q(\gamma')\right]^\sigma = 2\mathrm{Re} \left\{ \frac{Q}{1-Q} \left[N - \frac{1-Q^N}{1-Q} \right] \right\} \quad .$$

If δ/b is moderately large and $\theta' \neq -\alpha, \pi + \alpha$, we retain only the term in N and obtain

$$(63) \quad J \approx \frac{N(\cos b\gamma' - \delta\gamma' \sin b\gamma' - 1)}{1 - \cos b\gamma' + \delta\gamma' \sin b\gamma' + (\gamma'\delta)^2/2} \quad , \quad \gamma' = k(\sin\theta' + \sin\alpha).$$

However, if $\theta' \to -\alpha, \pi + \alpha$, then $\gamma' \to 0$ and all terms must be retained in J of (62). In this case,

$$\frac{Q}{1-Q} \to i \, \frac{(1 - b\gamma')}{(\delta + b)\gamma'} \quad , \quad \frac{1-Q^N}{1-Q} \to 1 + \frac{i\gamma' N(N+1)(\delta+b)}{2}$$

and J reduces to

(64) $$J = N(N-1)$$

as was expected from physical considerations.

On adding our present result for I_2 to I_1 of (57), we get

$$\langle|\psi|^2\rangle = I_1 + I_2$$

(65)
$$\approx \rho \int |\bar{\psi}_0(y')|^2 \left\{ \frac{(\gamma'\delta)^2/2}{1 - \cos b\gamma' + \delta\gamma' \sin b\gamma' + (\gamma'\delta)^2/2} - \mathrm{Re}\left[\frac{2Q(1-Q^N)}{N(1-Q)^2} \right] \right\} dy'$$

where $|\bar{\psi}_0(y')|^2 = 2|\bar{G}(\varphi', -\alpha)|^2/\pi k r'$. For a finite range $2d \ll r$, we have, to a first approximation,

$$\langle|\psi|^2\rangle \approx |\bar{\psi}_0(\varphi)|^2 \left\{ \frac{N(\delta\gamma)^2/2}{1 - \cos b\gamma + \delta\gamma \sin b\gamma + (\delta\gamma)^2/2} \right\}, \quad \varphi \neq -\alpha, \pi + \alpha;$$

(66)

$$\langle|\psi|^2\rangle \approx |\psi_0|^2 N^2, \quad \varphi \approx -\alpha, \pi + \alpha; \quad \gamma = k(\sin\varphi + \sin\alpha);$$

here $|\bar{\psi}_0(\varphi)|^2 = 2|\bar{G}(\varphi, -\alpha)|^2/\pi k r$ is the intensity corresponding to an element at the center of the range.

The function in the curly brackets of (66) is essentially the result obtained originally by Zernike and Prins[1]. Their basic paper contains plots of this expression as a function of γ and corresponding plots of $p(u) = \sum p_\sigma(u)$ of (11) and (12) as a function of u for several values of δ/b. Their curves for $\delta = \bar{b} - b = b$ show that $p(u)$ differs little from a step function except near $u = b$ and that the corresponding intensity

curve is a slowly varying function of Υ ; this is the 'more random' case. On the other hand, for $\delta/b = 9$, their graphs of p and of the intensity show sharp, almost equally-spaced maxima which decrease with the increasing argument; this is the 'more ordered' case. (Note that the value they give for the intensity corresponding to the specular angles is incorrect; they employed the limiting form of the expression in curly brackets, i.e., $N\delta^2/(b + \delta)^2$, rather than the much larger quantity N^2 obtained by taking the 'end effects' into account in the sums.) Their results were obtained numerically, but the behavior of $\langle|\psi|^2\rangle$ for $2d \ll r$, at least for the limiting cases, can be deduced directly from the limiting forms of $1 + J/N$ of (62).

Thus for $\delta/b \approx \bar{b}/b \gg 1$, we have

$$2\mathrm{Re}\left[\frac{Q}{1-Q}\right] \approx -2\,\frac{\sin(b\Upsilon)}{\bar{b}\Upsilon}\ , \quad -\frac{2}{N}\,\mathrm{Re}\left[\frac{Q(1-Q^N)}{(1-Q)^2}\right] \approx (N-1)\,\frac{\sin^2(d\Upsilon)}{(d\Upsilon)^2}\ ;$$

here we used $N\delta/2 \approx N/2\rho = d$. Hence (65) reduces to

$$(67)\quad \langle|\psi|^2\rangle \approx N|\bar{\psi}_o(\theta)|^2\left[1 + (N-1)\,\frac{\sin^2 d\Upsilon}{(d\Upsilon)^2} - \frac{2\sin b\Upsilon}{\bar{b}\Upsilon}\right]\ ,\quad \bar{b} \gg b,\ 2d \ll r\ ,$$

which could have been obtained directly without using $p_\sigma(u)$. Only the second term of (67) is important at $\theta = -\alpha,\ \pi + 2$; only the second and third are important in the vicinity of these angles; only the first is required when not in the vicinity of these angles. In general, since the second term drops off much more rapidly than the third as Υ increases, the third is significant over a larger range of $\theta - \alpha$. Thus for $|\theta - \alpha| > \lambda/2d$ (the first zero of the second term), we retain only the first and third terms to obtain a series of maxima and minima whose magnitudes hinge on $2b\sqrt{b}$, i.e., on the ratio of the exclusion width to the average spacing. The first maxima should be significant even for relatively large $2b\sqrt{b}$ (although for this case (67) is no longer realistic); here the pronounced 'main lobes' at the specular angles will be flanked by two fairly distinct smaller 'side lobes' .

On the other hand, if $\delta/b \to 0$, then

$$\frac{Q}{1-Q} \to \frac{i\, e^{ib\delta/2}}{2 \sin (b\gamma/2)} , \quad \frac{1-Q^N}{1-Q} \to e^{ib\gamma(N-1)/2} \quad \frac{\sin(b\gamma N/2)}{\sin(b\gamma/2)} .$$

Consequently (65) reduces to

$$(68) \quad \langle|\psi|^2\rangle \to |\bar{\psi}_o(\theta)|^2 \quad \frac{\sin^2(Nb\gamma/2}{\sin^2(b\gamma/2)} , \quad \bar{b} \to b, \quad 2d \ll r,$$

which is of the form usually obtained for a finite grating.

Eqs. (67) and (68) were derived for $2d \ll r$; however, we are primarily interested in the range $d \to \infty$. We extend the above results to $d \to \infty$ by simply using (67) and (68) with θ and r replaced by θ' and r' as the kernels of the integral $(2d)^{-1} \int_{-\infty}^{\infty} f(y')dy'$. Thus for $\delta/b \gg 1$, $d \to \infty$, the leading term of (67) yields I_1 of (57). To transform the remaining terms we introduce $\gamma' = k(\sin\theta' + \sin\alpha)$ as the integration variable and make use of $d\gamma' = -k \cos^2\theta' dy'/r'$ to write

$$(69) \quad I_2 \approx \frac{2\rho}{\pi k^2} \int_{-k(1-\sin\alpha)}^{k(1+\sin\alpha)} \left[\frac{|\bar{G}(\theta', -\alpha)|^2}{\cos^2\theta'}\right] \left\{ (N-1) \frac{\sin^2 d\gamma'}{(d\gamma')^2} - 2\rho \frac{\sin b\gamma'}{\gamma'} \right\} d\gamma'$$

$$\approx |\bar{\psi}|^2 \left\{ 1 - \frac{\mathrm{Si}[kb(1+\sin\alpha)] + \mathrm{Si}[kb(1-\sin\alpha)]}{\pi} \right\},$$

where we treated the function in square brackets in the integral as slowly varying and replaced it by its value for $\gamma' = 0$. The Si functions are tabulated in ref. 15, pp. 3-9. The first term of (69) equals the result obtained for a uniform distribution, i.e., (59); the remaining terms, which arise from the existence of a minimum spacing, may in general be neglected.

If $\delta/b \to 0$, $d \to \infty$, then using (68) and following essentially the same procedure as above, we obtain

$$\langle|\psi|^2\rangle \approx \left(\frac{2}{kb}\right)^2 \frac{1}{\pi N}\int_{-kb(1-\sin\alpha)/2}^{kb(1+\sin\alpha)/2} \left[\frac{|\bar{G}(\theta',-\alpha)|^2}{\cos^2\theta'}\right] \frac{\sin^2 N\beta}{\sin^2\beta}\, d\beta$$

$$(70) \quad \approx \left(\frac{2}{kb}\right)^2 \sum_m \frac{|\bar{G}(\theta_m,-\alpha)|^2}{\cos^2\theta_m} \equiv \sum_m |\bar{\psi}_m(\theta_m)|^2 \ ,$$

$$\theta_m = \sin^{-1}\left[\frac{2\pi m}{kb} - \sin\alpha\right], \quad |m| \le \frac{kb(1 \pm \sin\alpha)}{2\pi} \ ,$$

where we replaced the slowly varying function in the integral by its values for the zeros of $\beta = bk(\sin\theta' + \sin\alpha)/2$. The present result is the usual result for the propagating modes of an infinite grating of spacing b except that we take into account multiple scattering; the angles θ_m are the usual diffraction angles.

5. The Average Energy Flux

In this Section we make use of the results of Sections 3 and 4 to obtain an expression for the average energy flux - the function which is of principal physical interest. We derive an energy theorem which is useful to determine the 'transmission coefficient' and to obtain approximate solutions which conserve energy; we also use the theorem to check the range of validity of the closed-form approximations of the preceding sections.

For a single configuration of cylinders we write the total field as

$$(71) \qquad \Psi = \psi_i + \sum \psi_s \ , \qquad \psi_i = \exp[ikr\cos(\theta + \alpha)] \ ,$$

where ψ_s, the wave scattered by cylinder s, is given by (3). The total time-averaged energy flux per unit area divided by the time-averaged incident energy flux per unit area may be written as

$$(72) \qquad \underline{S} = \mathrm{Re}\left[\frac{\Psi^* \nabla \Psi}{ik}\right] \ .$$

$\underline{S}$ corresponds to the normalized Poynting flux $\mathrm{Re}[\underline{E} \times \underline{H}^*/2]$ for either polarization component (i.e., for either $\underline{E}$ or $\underline{H}$ parallel to the cylinders' axes);

$\underline{S}$ also corresponds to the normalized acoustic intensity $\mathrm{Re}(\underline{V}\, p^*/2)$, where $\underline{V}$ is the particle velocity and p is the excess pressure.

We have in general

$$(73) \qquad \nabla \Psi = \nabla(\psi_i + \sum \psi_s) = ik\, \psi_i\, \hat{n}_+ + \nabla \sum \psi_s \quad ,$$

where $\hat{n}_+ \equiv \hat{i} \cos\alpha - \hat{j} \sin\alpha$ is the unit vector along the incident wave normal; hence we may write

$$(74) \qquad \underline{S} = \mathrm{Re}\left[(\psi_i + \sum \psi_t)^* \left(\psi_i\, \hat{n}_+ + \frac{\nabla}{ik} \sum \psi_s\right)\right] \quad .$$

The energy flux averaged over the ensemble of configurations may be written as

$$(75) \qquad \langle \underline{S} \rangle = \mathrm{Re}\left[\hat{n}_+ + \psi_i^* (\hat{n}_+ + \nabla/ik) \sum \langle \psi_s \rangle + \sum\sum \langle \psi_t^* (\nabla_s/ik) \psi_s \rangle\right] \quad ;$$

this is permissible since the integrations in ψ_s commute with the gradient operation, and also since the gradient is invariant to a translation. The term $\nabla \sum \langle \psi_s \rangle$ may be obtained directly from the result of Section 3. Thus if we rewrite (23) as

$$(76) \qquad \sum \langle \psi_s \rangle \equiv \bar{\psi}_{\pm} = \bar{g}_{\pm}\, e^{ik(\pm x \cos\alpha - y \sin\alpha)} \quad ,$$

$$\bar{g}_{\pm} \equiv \frac{2\rho}{k \cos\alpha} \left\{ \begin{matrix} \bar{G}(-\alpha, -\alpha) \\ \bar{G}(\pi+\alpha, -\alpha) \end{matrix} \right\} , \left\{ \begin{matrix} x > 0 \\ x < 0 \end{matrix} \right\} \quad ,$$

we get

$$(77) \qquad (\nabla/ik) \sum \langle \psi_s \rangle = \bar{\psi}_{\pm} (\pm \hat{i} \cos\alpha - \hat{j} \sin\alpha) \equiv \bar{\psi}_{\pm}\, \hat{n}_{\pm} \quad ,$$

where $\hat{n}_-$ is the unit vector in the specular direction of reflection. However, the remaining terms of (75) are in general more difficult to treat because of their marked dependence on the degree of order of the distribution; for simplicity we consider only the limiting cases.

If $\delta/b \to 0$ (the limiting case of a grating) we deduce from (68) that

$$(78) \qquad \sum_t \sum_s \langle \psi_t^* (\nabla_t/ik)\psi_s \rangle = \sum_m |\bar{\psi}_m(\varrho_m)|^2 \hat{n}_m \ ;$$

here $\hat{n}_m = \hat{i} \cos \varrho_m + \hat{j} \sin \varrho_m$ are the unit vectors in the directions of the propagating mode normals. On the other hand, if $\delta/b \gg 1$, then we make use of the results for the uniform distribution as in (57) and (59) to obtain

$$\sum_t \sum_s \langle \psi_t^* (\nabla_s/ik)\, \psi_s \rangle = \sum_t \sum_s{}' \langle \psi_t^* (\nabla_s/ik)\, \psi_s \rangle + \sum_s \langle \psi_s^* (\nabla_s/ik)\, \psi_s \rangle$$

$$(79) \qquad \approx \bar{\psi}_{\pm}^* (\nabla/ik)\, \bar{\psi}_{\pm} + \sum \langle |\psi_s|^2 \hat{r}_s \rangle$$

$$= |\bar{g}_{\pm}|^2 \hat{n}_{\pm} + \frac{2\rho}{\pi k} \int \frac{|\bar{G}(\varrho', -\alpha)|^2}{r'} \hat{r}'\, dy' \ ,$$

where $\hat{r}_s = \hat{i} \cos \varrho_s + \hat{j} \sin \varrho_s$ is a unit vector from cylinder s to the observation point, and where $\hat{r}' = \hat{i} \cos \varrho' + \hat{j} \sin \varrho'$ is a unit vector from a point $(x = 0, y = y')$ on the plane of the distribution to the observation point. Here we kept only the term to the order of $(kr_s)^{-1/2}$ in performing the gradient operation in $\langle \psi_s^* (\nabla_s/ik)\psi_s \rangle$, so that the last term is correct to the order of $(kr)^{-1}$. This introduces no error in regard to energy considerations. For simplicity we restrict further discussion to the case where (79) applies.

We now substitute (76), (77), and (79) into (75) and write the average for a uniform distribution as

$$\langle \underline{S}_+ \rangle = \mathrm{Re}\left[|1 + \bar{g}_+|^2 \hat{n}_+ + \sum \langle |\psi_s(\varrho_s)|^2 \hat{r}_s(\varrho_s) \rangle \right] \ , \ x > 0 \ ;$$

(80)

$$\langle \underline{S}_- \rangle = \mathrm{Re}\left[\hat{n}_+ + \psi_i \bar{\psi}_-^* (\hat{n}_+ + \hat{n}_-) + |\bar{g}_-|^2 \hat{n}_- + \sum \langle |\psi_s(\pi - \varrho_s)|^2 \hat{r}_s(\pi - \varrho_s) \rangle \right] , \quad x < 0 \ ,$$

where $\hat{r}_s(\varrho_s) = \hat{i} \cos \varrho_s + \hat{j} \sin \varrho_s$, and $|\varrho_s| \leq \pi/2$. The terms in $\hat{n}_{\pm}$ and $\hat{r}_s$ are

the 'coherent' and 'incoherent' components respectively. The coherent component of the first line of (80) is the transmitted flux $|\psi_i + \bar{\psi}_+|^2$, and that of the second line consists of the incident and reflected fluxes plus a term arising from interference of the incident and reflected waves. The interference term $\psi_i^* \bar{\psi}_-(\hat{n}_+ + \hat{n}_-)$ is equal to $-\hat{j}2\bar{g}_-$ sin α cos$(2kx \cos\alpha)$; since this term is parallel to the plane of the distribution, and since it is also a rapidly oscillating function of x, it may be ignored except for $\alpha \to \pi/2$. If $\alpha \to \pi/2$, then (from Section 3 and 4) $\bar{g}_{\pm} \to -1$, while the terms containing $\hat{r}_s$ vanish; consequently $\langle \underline{S}_{\pm} \rangle \to 0$ as $\alpha \to \pi/2$.

From the above we derive an energy theorem valid for relatively arbitrary scatterers. In order to satisfy conservation of energy when a wave is scattered by a planar configuration of non-absorbing scatterers we require that the component of $\underline{S}$ normal to the plane be continuous. Since this must hold for each configuration, it must also hold for the ensemble average. We may therefore equate $\langle \underline{S}_+ \cdot \hat{i} \rangle$ of (80) with $\langle \underline{S}_- \cdot \hat{i} \rangle$ to obtain

$$(81)\quad |1 + \bar{g}_+|^2 \cos\alpha + \sum \langle |\psi(\theta_s)|^2 \cos\theta_s \rangle = (1 - |\bar{g}_-|^2)\cos\alpha - \sum \langle |\psi_s(\pi - \theta_s)|^2 \cos\theta_s \rangle;$$

i.e., the normal component of the transmitted flux must equal the difference of the normal components of the incident and reflected fluxes.

In order to simplify the above we express the terms in cos θ_s as

$$\frac{2\rho}{\pi k}\int_{-\infty}^{\infty} \left[|\bar{G}(\theta', -\alpha)|^2 + |\bar{G}(\pi - \theta', -\alpha)|^2 \right] \frac{\cos\theta'}{r'}\, dy' .$$

We now change the integration variable to $\theta' = \sin^{-1}[(y - y')/r']$, $d\theta' = -\cos\theta' dy'/r'$, and obtain

$$(82)\quad \frac{2\rho}{\pi k}\int_{-\pi/2}^{\pi/2} \left[|\bar{G}(\theta', -\alpha)|^2 + |\bar{G}(\pi - \theta', -\alpha)|^2 \right] d\theta' = \frac{2\rho}{\pi k}\int_0^{2\pi} |\bar{G}(\theta', -\alpha)|^2 d\theta' \equiv \rho Q \equiv \rho \int_0^{2\pi} q(\theta', -\alpha) d\theta' .$$

Here Q, the average 'scattering cross section' per unit length of cylinder, is the average power scattered to infinity per unit length of cylinder divided by the incident energy density. ρQ is the 'scattering cross section per unit area of dis-

tribution', and $\rho q(\theta')$ is the 'differential scattering cross section per unit area' (i.e., the power scattered into unit solid angle by unit of distribution). More generally we have

$$(83) \qquad Q \equiv \mathrm{Re}\left[\frac{1}{ik}\int_0^{2\pi} <\psi_o^*(\theta)\,\frac{\partial\psi_o(\theta)}{\partial r}> r d\theta\right] .$$

Because of (82) we may rewrite (81) as

$$(84) \qquad -2\mathrm{Re}\,\bar{g}_+ - |\bar{g}_+|^2 = |\bar{g}_-|^2 + \rho Q \sec\alpha\ ;$$

here $\rho \sec\alpha$ is the average number of scatterers 'illuminated' by unit area of incident wave. The theorem of (84) reads as follows: on the average, the flux per unit area arising from interference of the incident wave with the coherent forward-scattered wave, plus the coherent scattered intensity in both the forward and specularly reflected directions, equals the negative of the scattering cross section of one scatterer times the number of elements excited by unit area of incident wave. (We note that if we set $Q = 0$, then without averaging (84) yields the corresponding result for a grating such that $kb < 1$, i.e., for a grating with only zero modes.)

To make the result of (84) self-evident we rewrite the first line of (80) as

$$(85) \qquad <\underline{S}_+> = T\hat{n}_+ + \rho\int_0^{\infty} q(\theta',-\alpha)\,\frac{\hat{r}'}{r'}\,dy', \quad T \equiv |1+\bar{g}_+|^2 = |\psi_i+\bar{\psi}_+|^2, \quad q = \frac{2}{\pi k}|\bar{G}(\theta',-\alpha)|^2 ,$$

where T is the 'transmission coefficient' and q is the differential cross section defined for (82). If we presuppose a directional receiver receiving radiation from only a finite (small compared to r') range of the distribution, then in the direction of transmission $\hat{n}_+$ we may ignore the term in r'^{-1}. In directions other than $\hat{n}_+$ we take only the term in r' into account. (Here we visualize the receiver as fixed to rotate around r, θ (or x, y); note $r'^2 = x^2 + (y - y')^2$.)

Comparison of T of (85) with eq. (84) makes the physical interpretation of the theorem clear: the left side of (84) is the energy 'missing' from unit area of the wave received in the direction of transmission; the right side is the energy reflected and scattered from unit area of incident wave. The equality follows

from the energy principle. (Note that the r'^{-1} term we neglected, i.e., the energy scattered <u>into</u> the receiver in the direction of transmission, could be dealt with by splitting Q into two parts.)

For greater utility we rewrite (84) in terms of the scattering amplitudes:

$$(86)\quad \frac{4\rho}{k\cos\alpha}\left\{\mathrm{Re}\ \bar{G}(-\alpha,-\alpha) + \frac{1}{2\pi}\int_0^{2\pi}|\bar{G}(\theta,-\alpha)|^2 d\theta + \frac{\rho}{k\cos\alpha}\left[|\bar{G}(-\alpha,-\alpha)|^2 + |\bar{G}(\pi+\alpha,-\alpha)|^2\right]\right\} = 0\ .$$

Since the integral in Sections 3 and 4 which yielded the above terms could have been evaluated by the method of stationary phase, we see that (86) holds for a uniform distribution of arbitrary identical cylinders (i.e., not necessarily circular), provided that $<G(\theta', -\alpha)>$ is a slowly varying function of y' compared to $\exp[ik(r'-y'\sin\alpha)]$. To lowest order in $\rho/k\cos\alpha$, (86) reduces to the well-known theorem for an <u>isolated</u> scatterer:

$$(86')\quad \mathrm{Re}\ C(-\alpha,-\alpha) = \mathrm{Re}\ C_o = -\frac{1}{2\pi}\int_0^{2\pi}|C(\theta,-\alpha)|^2\, d\theta \equiv -\frac{k}{4}Q^1 \equiv -\frac{k}{4}\int_0^{2\pi} q^1(\theta,-\alpha)\, d\theta\ .$$

Theorem (86) relates the forward scattered amplitude of an isolated non-absorbing scatterer to its scattering cross section Q^1.

From (86') we see that a single-scattering approximation that satisfies the physical requirement that energy be conserved can be obtained only by neglecting all powers of $\rho/k\cos\alpha$ higher than the first in (80) and (85). Thus the correct single-scattered approximation of (85), say $<\underline{S}_+^1>$, is obtained by replacing T and q by

$$(87)\quad T^1 = 1 - \frac{4\rho}{k\cos\alpha}\ \mathrm{Re}\ C_o = 1 - \frac{\rho Q^1}{\cos\alpha}, \qquad q^1(\theta', -\alpha) = \frac{2}{\pi k}|C(\theta',-\alpha)|^2\ .$$

From T^1 we see that the energy 'missing' from unit area of incident wave is, to lowest order of $\rho/k\cos\alpha$, simply the single-scattered energy radiated to infinity. Note that not all the single-scattered terms have been retained in (87); i.e., we must neglect $4(\rho/k\cos\alpha)^2|C_o|^2$ to insure conservation of energy. The above is in accord with our results of Section 3 and 4; i.e., for the 'more random' distributions ($\bar{b} = \rho^{-1} \gg b$) such that $bk \gg 1$, we may neglect multiple scattering except for $\alpha \to \pi/2$. (It is tacitly assumed for a 'uniform distribution' that $\bar{b} \gg b$, i.e., that the average separation of scatterers is large compared to their minimum separation.)

If $\alpha \to \pi/2$, then (87) breaks down; here we use the results of Sections 3 and 4:

$$\bar{G}(\theta, -\alpha) \to \frac{C(\theta, -\alpha)}{1 - C_o \bar{U}_-} \quad , \quad \bar{U}_- \to \frac{2\rho}{k \cos \alpha} \quad , \quad \alpha \to \pi/2 \quad .$$

Substituting these values into (86) shows that the equality is satisfied, i.e., our closed-form approximations for the multiple-scattered values satisfy conservation of energy for $\alpha \to \pi/2$.

The theorem of (84) can be readily extended to absorbing cylinders by replacing Q in (84), etc., by $Q + Q_a$, where Q_a, the 'average absorption cross section', is defined as the average power flow into unit length of cylinder divided by the incident energy density, i.e., the fraction of the incident energy discipated as heat by unit length of cylinder. We have in general

$$(88) \quad Q_a = -\mathrm{Re} \frac{1}{ik} \int_o^{2\pi} \left\langle \Psi^* \frac{\partial \Psi}{\partial r_s} \right\rangle r_s d\theta_s = \mathrm{Re} \frac{KM}{ik} \int_o^{2\pi} \left\langle \Phi_s^* \frac{\partial \Phi_s}{\partial r_s} \right\rangle \Big|_{r_s=a} a d\theta_s \quad ,$$

where Φ_s, the field within cylinder s, was defined in (4'); the above equality follows from the fact that Q_a is defined for all values of r_s including $r_s = a$ and from the boundary conditions $\Psi = M \Phi_s$, $\frac{\partial \Psi_s}{\partial r_s} = K \frac{\partial \Phi_s}{\partial r_s}$ at $r_s = a$. The conclusions we reached for the case of non-absorbing cylinders can be carried over intact to absorbing cylinders. (Note that $\rho \sec \alpha \, Q_a$ is to be added to the left side of (86) and that (86') is to be replaced by $\mathrm{Re}\, C_o = -(k/4)(Q^1 + Q_a^1)$.)

We now show that our results for isotropic scatterers conserve energy for all values of α. From (32') we have

$$(89) \quad \bar{G} = A_o/(1 - A_o X) \quad , \quad X = \bar{U}_+ + \bar{U}_- = 2\rho/k \cos \alpha \quad .$$

We substitute (89) into (86) and add $Q_a \rho \sec \alpha$ to the left side of (86) to take into account absorption. We obtain initially

$$(90) \quad \frac{Q_a \rho}{\cos \alpha} + \frac{4\rho}{k \cos \alpha} \left\{ \mathrm{Re}\bar{G} + |\bar{G}|^2 \left(1 + \frac{2\rho}{k \cos \alpha} \right) \right\} = \frac{X}{2} \left\{ Q_a + \frac{4}{k} \left[\mathrm{Re} \frac{A_o}{1 - A_o X} + \frac{|A_o|^2}{|1 - A_o X|^2} (1 + X) \right] \right\} = 0.$$

From the single-scattered theorem (86') with Q^1 replaced by $Q^1 + Q_a^1$ we find $\mathrm{Re}\, A_o = -(k/4)(Q^1 + Q_a^1) = -|A_o|^2 - (k/4)Q_a^1$. Consequently the leading term in the square brackets of (90) may be rewritten as

$$\mathrm{Re}\,\frac{A_o}{1 - A_o X} = \mathrm{Re}\,\frac{(A_o - |A_o|^2 X)}{|1-A_o X|^2} = -\frac{k(Q^1 + Q_a^1 + Q^1 X)}{|1-A_o X|^2} \quad .$$

Substituting this result into (90) and remembering that $(k/4)Q^1 = |A_o|^2$, we obtain

$$(90') \qquad \frac{X}{2}\left[Q_a - \frac{Q_a^1}{|1-A_o X|^2} \right] = 0 \quad .$$

From (4') and (88) we find for isotropic scatterers

$$Q_a = -\mathrm{Re}\left[\frac{k'MK}{ik} J_o(k'a)\, J_o'(k'a)\, 2\pi \langle |D_o|^2 \rangle \right] = \frac{Q_a^1 |\bar{G}|^2}{|A_o|^2} \quad ,$$

where we make use of $D_o = E_o \bar{G}/A_o$; see discussion of (4'). Substituting this result into (90') shows that the left side vanishes.

6. Distributions of Bosses on a Perfectly Reflecting Plane

In this section we extend the results of the preceding sections to the analogous problem of scattered reflection from semicylindrical bosses on a perfectly reflecting plane as in Fig. 4. We write the total average wave as

$$(91) \qquad \bar{\Psi} = \psi_p + \bar{\psi}_r \quad ,$$

where

$$(92) \qquad \psi_p = \pm\, \psi_i(\varrho, \pi - \alpha) = \pm \exp[-ikr \cos(\varrho - \alpha)]$$

is the incident wave (coming from the right in Fig. 3). The plus sign in (92) is taken for the boundary conditions $\frac{\partial \bar{\Psi}}{\partial x} = 0$ on the plane, and the minus sign is for $\bar{\Psi} = 0$ on the plane. The function $\bar{\psi}_r$, the average reflected wave, may be written as

(93) $$\bar{\psi}_r = \psi_i + \bar{\psi}_{\parallel,\perp} \quad , \quad \psi_i = \psi_i(\theta, -\alpha) = \exp\left[ikr \cos(\theta + \alpha)\right] ,$$

where ψ_i, the wave reflected from the plane in the absence of the bosses, is the incident wave we used in the previous sections. The scattered component $\bar{\psi}_{\parallel,\perp}$ is obtained from the result of Sec. 3 by means of the image technique[17] illustrated in Fig. 3.

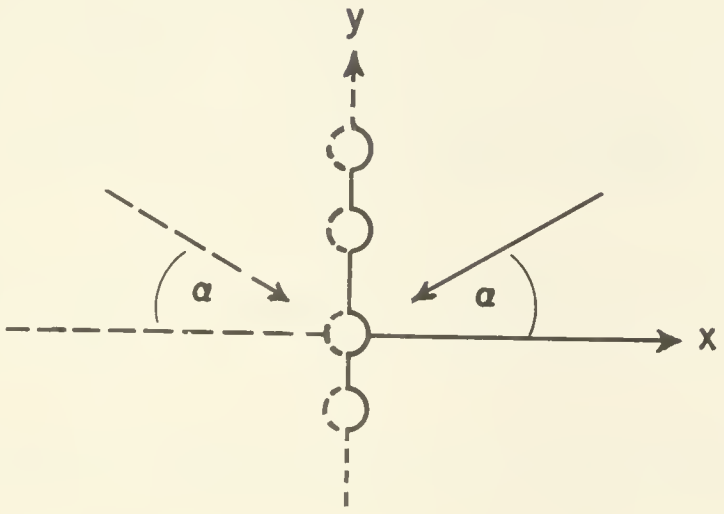

Fig. 3 Plane wave incident on a confuguration of bosses on a perfectly reflecting plane.

For the component polarized parallel to the plane of incidence (perpendicular to the axes of the bosses), or for a rigid plane in acoustics, we have

(94) $$\bar{\psi}_{\parallel} = \bar{\psi}(\theta, -\alpha) + \bar{\psi}(\theta, \pi + \alpha) ,$$

where $\bar{\psi}(\theta, -\alpha)$ is the result of Sec. 3; here the boundary condition on the plane is $\frac{\partial \bar{\psi}}{\partial x} = 0$. For the component polarized perpendicular to the plane of incidence, or for a free surface in acoustics, we have

(95) $$\bar{\psi}_{\perp} = \bar{\psi}(\theta, -\alpha) - \bar{\psi}(\theta, \pi + \alpha) ;$$

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17. See Rayleigh, Phil. Mag. 14, 350(1907) for a treatment of the single cylinder.

here the boundary condition on the plane is $\bar{\Psi} = 0$.

For simplicity we now write the result of Section 3 in the form

$$(96) \qquad \bar{\psi} = \bar{U}\,\bar{G}, \quad \bar{U} = \frac{2\rho}{k \cos\alpha}\,\psi_i, \quad \bar{G} = C + \bar{Z};$$

i.e, we have only a single value of $\bar{U}$, since only the case of $x > 0$ of (23) is of interest; also we have split the scattering amplitude into a single-scattered and a multiple-scattered component. In terms of (96) we rewrite (94) and (95) as

$$\bar{\psi}_{\parallel,\perp} = \bar{U}\left[\bar{G}(\varphi, -\alpha) \pm \bar{G}(\varphi, \pi + \alpha)\right]_{\varphi = -\alpha} \equiv \bar{U}\,\bar{G}_{\parallel,\perp},$$

or as

$$(94') \qquad \bar{\psi}_{\parallel} = \bar{U}\,2(C_{\parallel} + \bar{Z}), \quad C_{\parallel} = \sum_{n=-\infty}^{\infty} A_n \cos^{2n}(\alpha + \pi/2),$$

and

$$(95') \qquad \bar{\psi}_{\perp} = \bar{U}\,2\,C_{\perp}, \quad C_{\perp} = \sum A_n' \sin^{2}n(\alpha + \pi/2),$$

where the multiple-scattered terms in $\bar{\psi}_{\perp}$ vanish to our present order of approximation because of the symmetry of $\bar{Z}$ with respect to the plane, i.e., because $\bar{Z}(\varphi,-\alpha)=\bar{Z}(\varphi,\pi+\alpha)$. The single-scattered coefficients $C_{\parallel,\perp}$ were obtained by means of

$$C_{\parallel,\perp} = \left[\frac{C(\varphi,-\alpha) \pm C(\varphi,\pi + \alpha)}{2}\right]_{\varphi = -\alpha},$$

where $C(\varphi, -\alpha)$ is given by (7). If the bosses are not circular, then we express $C_{\parallel,\perp}$ in terms of surface integral).

From Section 3 we see that $\bar{\psi}_{\parallel} \to -2e^{-iky}$ as $\alpha \to \pi/2$; on the other hand, from $C_{\perp}$ of (95') we see that $\bar{\psi}_{\perp} \to 0$ as $\alpha \to \pi/2$. Substituting these values into (93) we find that $\bar{\psi}_r \to \mp e^{-iky} = -\psi_p$; consequently the total field $\psi_p + \bar{\psi}_r$ vanishes as $\alpha \to \pi/2$, regardless of polarization. Thus as grazing incidence is approached the average reflected values of both $\underline{E}$ and $\underline{H}$ approach the negative of the incident value. (Note that if the bosses were absent, then only the component of $\underline{E}$ perpendicular to the plane of incidence would show the phase change.) Furthermore, since the variance of ψ vanishes for $\alpha \to \pi/2$, this result holds even for a single configuration.

Hence we deduce that a planar striated surface should act like a perfect

reflector as grazing incidence is approached; this behavior is well known experimentally and may be easily verified for an arbitrary rough planar surface. In addition the phase change of <u>both</u> polarization components is in accord with interference experiments for even highly conducting surfaces; the phase change of the parallel component is usually interpreted on the basis of a small but non-vanishing dielectric constant* - we see that it may also be accounted for on the basis of multiple scattering. Thus the distribution behaves like a dielectric plane as $\alpha \to \pi/2$.

To our present order of approximation (i.e., the assumption $kb \gg 1$), the perpendicular component shows no multiple-scattering effects. However, if we assume $ka \ll 1$ and retain only the coefficients B_{os}, $B_{\pm 1s}$ in (4) and use these to determine $\bar{\psi}_\perp$, we find (see Appendix I)

$$\bar{\psi}_\perp = \bar{U} \frac{4A_1 \cos^2 \alpha}{1 - 2A_1 \bar{X}'}, \quad \bar{X}' = 2\rho \int_0^\infty \frac{H_1(ku)}{ku} \cos(ku \sin \alpha) p(u) du.$$

Comparing $\bar{X}'$ with $\bar{X} = \bar{U}_+ + \bar{U}_-$ of (32) and (32'), we deduce that the effects of multiple scattering may be neglected in $\bar{\psi}_\perp$ even for $\alpha \to \pi/2$.

The intensities corresponding to (94'), (95') may be written operationally as

$$(97) \quad \langle |\bar{\psi}_\parallel|^2 \rangle = 4 \langle |U|^2 \rangle |C_\parallel + \bar{Z}|^2, \quad \langle |\psi_\perp|^2 \rangle = 4 \langle |U|^2 \rangle |C_\perp|^2,$$

where the operators $\langle |U|^2 \rangle$ may be obtained by inspection of the results of Sec. 4. The first reduces to 4 and the second to 0 as $\alpha \to \pi/2$; however, the total intensities $\langle |\psi_i + \psi_p + \psi_{\parallel,\perp}|^2 \rangle$ vanish for either case.

The average normalized energy flux for either polarization component is obtained essentially as in Sec. 5. Thus proceeding as for (74), (75) we find

$$(98) \quad \langle \underline{S} \rangle = \mathrm{Re} \langle (\psi_p + \psi_i + \sum \psi_t)^* (\psi_p \hat{n}_- + \psi_i \hat{n}_+ + \frac{\nabla}{ik} \sum \psi_s) \rangle$$

where $\hat{n}_\pm = \pm \hat{i} \cos \alpha - \hat{j} \sin \alpha$ are unit vectors and where ψ_t (or ψ_s) is the wave

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* The non-vanishing dielectric constant of a 'uniform medium' must ultimately be interpreted on the basis of elementary scatterers. The present analysis is one approach; that of Born[5] (dipole scatterers) is another.

scattered by boss t (or s). For brevity we now define $\bar{F}(\theta, \pi+\alpha) \equiv G(\theta,-\alpha) \pm G(\theta,\pi+\alpha)$. Using the results of (76) to (79) for $x > 0$, e.g., $\sum \langle \psi_t \rangle = \bar{f}\, \psi_i$, $\bar{f} = (2\rho/k \cos\alpha)\, \bar{F}(-\alpha, \pi+\alpha)$, etc., we proceed as for (80) and obtain for a uniform distribution

$$(99) \quad \langle \underline{S} \rangle = \mathrm{Re}\left\{ \hat{n}_- + 2\cos(2kx \sin\alpha)(1+\bar{f})(\hat{n}_- + \hat{n}_+) + |1+\bar{f}|^2 \hat{n}_+ + \sum \langle |\psi_s|^2 \hat{r}_s \rangle \right\},$$

where $\hat{r}_s = \hat{i}\cos\theta_s + \hat{j}\sin\theta_s$. Eq. (99) consists of essentially four terms: the incident flux(in $\hat{n}_-$); the flux arising from interference of the incident and reflected waves (in $\hat{n}_- + \hat{n}_+$); the reflected flux (in $\hat{n}_+$); the incoherently scattered flux (in $\hat{r}_s$). The last two terms comprise the total reflected flux, say $\langle \underline{S}_r \rangle$; note that $\langle \underline{S}_r \rangle$ is formally identical with $\langle \underline{S}_+ \rangle$ of (82). It can be seen that $\langle \underline{S}_r \rangle \to 0$ as $\alpha \to \pi/2$.

From (99) we derive a reflection theorem analogous to the transmission theorem of (84). If the bosses are non-absorbing, then the component of $\langle \underline{S} \rangle$ normal to the plane they lie on must vanish in order that energy be conserved. Consequently we may equate $\langle \underline{S} \cdot \hat{i} \rangle$ to zero to obtain

$$(100) \qquad -(2\,\mathrm{Re}\,\bar{f} + |\bar{f}|^2)\cos\alpha = \sum \langle |\psi_s|^2 \cos\theta_s \rangle ,$$

which is simple compared to (83). Proceeding as for (81) we rewrite the right side of (100) as

$$(101) \quad \frac{2\rho}{\pi k}\int_{-\infty}^{\infty} |\bar{F}(\theta', \pi+\alpha)|^2 \frac{\cos\theta'}{r'}\, dy' = \frac{2\rho}{\pi k}\int_{-\pi/2}^{\pi/2} |\bar{F}(\theta', \pi+\alpha)|^2\, d\theta' \equiv \rho P \equiv \rho \int p(\theta')\, d\theta',$$

where P is the scattering cross section per unit length of boss and p is the differential scattering cross section. Hence (100) reduces to

$$(102) \qquad -2\,\mathrm{Re}\,\bar{f} - |\bar{f}|^2 = P\rho \sec\alpha$$

(cf. (84)). This theorem states that <u>on the average, the flux (per unit area) arising from interference of the wave reflected from the plane and the average coherently scattered wave plus the coherently scattered intensity equals the negative of the scattering cross section of one boss times the number of bosses excited by unit area of incident wave.</u>

To make (102) self-evident, we proceed as for (86) and obtain the same formal result. We write the reflected flux (the last two terms of (99)) received by a directional receiver at r, θ as

$$(103)\qquad \langle \underline{S}_r \rangle = R\,\hat{n}_+ + \rho\int_{-\infty}^{\infty} p(\theta')\,\frac{\hat{r}'}{r}\,dy', \quad R = |1+\bar{f}|^2, \quad p = \frac{2}{\pi k}\,|\bar{F}(\theta,\pi+\alpha)|^2 .$$

Here R is the 'reflection coefficient'. We again presuppose a directional receiver and neglect the $\hat{r}'$ term in the direction $\hat{n}_+$; for directions other than $\hat{n}_+$ we take only the $\hat{r}'$ term into account. See discussion of (86).

Comparison of R of (103) with (102) makes the physical interpretation of the theorem clear: the left side of (102) is the energy 'missing' from unit area of the wave that would be reflected from the plane in the absence of the bosses; the right side is the energy scattered from unit area of the incident wave. The equality follows from the energy principle.

We rewrite (102) in terms of the scattering amplitude as

$$(102')\qquad \frac{4\rho}{k\cos\alpha}\left\{\mathrm{Re}\,\bar{F}(-\alpha,\pi+\alpha) + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}|\bar{F}(\theta,\pi+\alpha)|^2 d\theta + \frac{\rho}{k\cos\alpha}\,|F(-\alpha,\pi+\alpha)|^2\right\} = 0$$

(cf.(86)) . To the lowest order in $\rho/k\cos\alpha$, (102') reduces to its single-scattered value:

$$(102'')\qquad \mathrm{Re}F^1(-\alpha,\pi+\alpha) = -\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}|F^1(\theta,\pi+\alpha)|^2\,d\theta = -\frac{k}{4}P^1 = -\frac{k}{4}\int p^1(\theta')d\theta';$$

here $F^1(\theta,\pi+\alpha) = [C(\theta,-\alpha) \pm C(\theta,\pi+\alpha)]/2$ is the scattering amplitude for an isolated boss and P^1 is its scattering cross section. From our derivations we see that (102'') applies for a boss of relatively arbitrary cross-sectional area on a perfectly reflecting plane; this theorem states that <u>the scattering cross section of the boss is proportional to the real part of the scattering amplitude in the direction of specular reflection of the plane</u>[18].

In applying the preceding results to arbitary scatterers, it is tacitly assumed that $\bar{F}$ is a slowly varying function of y' compared to $\exp[ikr' - iky'\sin\alpha]$. However we may obtain a corollary of (102'') directly from (98) which places no restriction on $\bar{F}$, viz.,

$$\mathrm{Re}\,\psi_i^*\int_{-\infty}^{\infty}\psi^1(y')dy' = -\frac{1}{2\cos\alpha}\int_{-\pi/2}^{\pi/2}|\psi^1(\theta)|^2\,r\,d\theta, \quad r\to\infty .$$

Here $\psi^1(y')$ is the wave scattered by an isolated boss at x = 0, y = y'. (Evaluation

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18. We have also derived this theorem directly for a single boss; see "Certain Transmission and Reflection Theorems," Math. Res. Grp., NYU, EM - 54.

of the integral over y' by the method of stationary phase yields the results of (102").) Using the present result together with the direct derivation of (102") in ref. 18 we obtain the equality

$$\psi_i^* \int_{-\infty}^{\infty} \psi^1(y')dy' = (2/k \cos \alpha)\, F^1(-\alpha, \pi+\alpha)$$

for an arbitrary boss; the analogous equality holds for the transmission problem.

On the basis of the above we now obtain a single-scattering approximation of (103), say $< \underline{S}_r^1 >$, which satisfies conservation of energy. Thus, keeping only terms of the order $\rho/k \cos \alpha$ in (103), we obtain in analogy to (87)

$$(104) \quad R^1 = 1 + \frac{4\rho}{k \cos \alpha} \operatorname{Re} F^1(-\alpha, \pi+\alpha) = 1 - \frac{\rho P^1}{\cos \alpha}, \quad p^1(\theta, \pi+\alpha) = \frac{2}{\pi k} |F^1(\theta, \pi+\alpha)|^2 .$$

Here we may rewrite the reflection coefficient as

$$(104') \qquad R^1 \equiv 1 - \sigma \approx e^{-\sigma}, \quad \sigma \equiv \rho P^1 \sec \alpha ,$$

and define σ to be the 'scattering cross section of the distribution per unit area of incident wave'.

For simplicity we assumed in the above that the bosses were non-absorbing; however, absorption may be taken into account by replacing P by $P + P_a$ where P_a is the absorption cross section; see Section 5.

We may employ (104) for the component polarized perpendicular to the plane of incidence for all values of α, provided that $\rho/k \ll 1$; here there are no 'infinities' since Re $F_{\perp}^1(-\alpha, \pi+\alpha)$ vanishes more rapidly than cos α as $\alpha \to \pi/2$. For the parallel component, however, we must take into account multiple scattering for $\alpha \to \pi/2$. Thus on the basis of the preceding Section we use

$$(105) \quad \overline{G}_{\parallel}(\theta, \pi+\alpha) = 2C_{\parallel}(\theta, \pi+\alpha) + \frac{2C(\theta, -\pi/2)C(-\pi/2, -\alpha)\overline{U}_-}{1 - C_o \overline{U}_-}, \quad \overline{U}_- \to \frac{2\rho}{k \cos \alpha}, \ \alpha \to \pi/2 .$$

We now apply our results to perfectly conducting circular bosses; here we have

$$(106) \quad C_{\parallel}(\theta, \pi+\alpha) = \sum A_n \cos n(\alpha+\pi/2)\cos n(\pi/2-\theta), \quad C_{\perp} = \sum A_n' \sin n(\alpha+\pi/2)\sin n(\pi/2-\theta),$$

$$A_n = -J_n'(ka)/H_n'(ka) , \qquad A_n' = -J_n(ka)/H_n(ka) .$$

Consequently (102'') yields the scattering cross sections

$$(107)\quad P_{\parallel}^{1} = \frac{8}{k}\sum |A_n|^2 \cos^2 n(\alpha+\pi/2)\ ,\quad P_{\perp}^{1} = \frac{8}{k}\sum |A_n'|^2 \sin^2 (\alpha+\pi/2)\ .$$

These results follow from either the left or right side of (102''), since

$$\mathrm{Re}\ A_n' = -\mathrm{Re}\ J_n/(J_n + iN_n) = -J_n^2/(J_n^2 + N_n^2) = -|A_n'|^2\ ,\ \text{etc.}$$

For $ka \to 0$ we obtain the reflection coefficients

$$(108)\quad R_{\parallel} \approx 1 - \frac{\pi^2(ka)^4}{k\bar{b}}\left[\frac{1+2\sin^2\alpha+3B^2(1-\sin\alpha)}{2\cos\alpha(1+B^2/\cos^2\alpha)}\right],\ R_{\perp}\approx 1-\frac{\pi^2(ka)^4}{k\bar{b}}\cos\alpha,\ B=\frac{\pi(ka)^2}{k\bar{b}},\ \bar{b}=\rho^{-1},$$

where the terms containing B are to be taken into account only for $\alpha \to \pi/2$.
Similarly we have the differential scattering cross section per unit area

$$(109)\quad \rho p_{\parallel} \quad \frac{\pi(ka)^4}{k\bar{b}}\left[\frac{(1+a\sin\alpha\sin\theta)^2+B^2 2(1-\sin\alpha)(1+\sin\theta)^2}{1+B^2/\cos^2\alpha}\right],\ \rho p_{\perp}\approx \frac{\pi(ka)^4}{2k\bar{b}}\cos^2\alpha\cos^2\theta\ .$$

In terms of the 'horizon angle' $\beta = \pi/2 - \alpha$, we see that as $\beta \to 0$ the back scattered values of (109) vanish like β^2 and β^4 respectively; this seems to agree qualitatively with measurements for the sea surface[19].

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19. D.E. Kerr, Propogation of Short Radio Waves (McGraw-Hill, N.Y., 1951) p. 481 ff.

APPENDIX

If only the coefficients $\bar{B}_0$ and $\bar{B}_{\pm 1}$ are retained in (25), then a closed-form representation for $\bar{G}(\phi, -\alpha) \approx \bar{B}_0 + B_1 e^{i\phi} + B_{-1} e^{-i\phi}$ may be obtained without further approximation. Introducing the operators

$$P_{\pm} \equiv \int_0^{\infty} du\, p(u) \left[e^{iku \sin \alpha} \pm e^{-iku \sin \alpha} \right]$$

and making use of the relations $H_0(x) - H_2(x) = 2H_1' = 2(H_0 - H_1/x)$, $H_0 + H_2 = 2H_1/x$, we find

$$B_0 + B_1 e^{i\phi} + B_{-1} e^{-i\phi} = \frac{2A_1 \cos \alpha \cos \phi}{1 - 2A_1 P_+\left(\frac{H_1}{ku}\right)}$$

(A.1)

$$+ \frac{A_0 - 2A_1 \sin \alpha \sin \phi + A_0 A_1 \left[(-1 + \sin \alpha \sin \phi) 2P_+(H_0)\right] + 2P_+\left(\frac{H_1}{ku}\right) + 2i(\sin\phi - \sin\alpha) P_-(H_1)\right]}{\left[1 - A_0 P_+(H_0)\right]\left[1 - 2A_1 P_+(H_1')\right] + 2A_0 A_1 \left[P_-(H_1)\right]^2}$$

(A.1) reduces to the same form as (28) if only the terms in $(ku)^{-\frac{1}{2}}$ are retained in the asymptotic expansion of the Hankel functions.

The essential difference between (A.1) and (28) is in regard to the term in cos ϕ; in (28) multiple scattering is neglected for the components not symmetrical to the plane.

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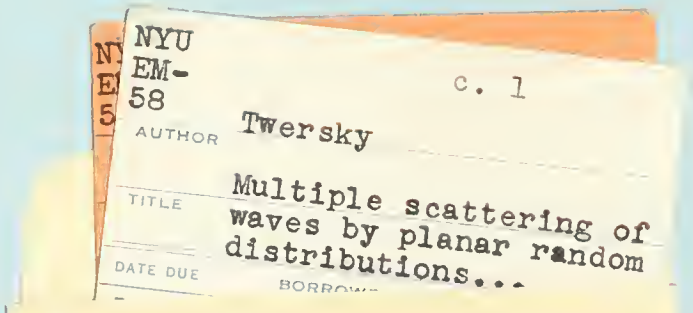

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Multiple scattering of waves by planar random distributions of parallel cylinders and bosses.